Math 4140: Homework 11

Due April 21, 2009

1. Find explicit $\mathfrak{sl}_2(\mathbb{C})$-module isomorphisms between

   (a) The trivial representation of $\mathfrak{sl}_2(\mathbb{C})$ and V_0,

   (b) The natural representation of $\mathfrak{sl}_2(\mathbb{C})$ and V_1,

   (c) The adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$ and V_2.

2. Let V_d be the d-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module. Let $v \in V_d$ be such that $e \cdot v = 0$, so that

$$\{v, f \cdot v, f^2 \cdot v, \ldots, f^{d-1} \cdot v\}$$

   is a basis for V_d.

   (a) Find $c_1 \in \mathbb{C}$ such that $e \cdot f \cdot v = c_1 \cdot v$.

   (b) Let $c_k \in \mathbb{C}$ be such that

$$e \cdot f^k \cdot v = c_k f^{k-1} \cdot v.$$

   Find a formula for c_k and prove that your formula is correct (Hint: Induction works well here).

3. Let V be a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module (not necessarily irreducible). Define

$$\begin{array}{rccc} C: & V & \longrightarrow & V \\ & v & \mapsto & \left(ef + fe + \frac{1}{2}h^2\right) \cdot v \end{array}$$

   (a) Show that C is an $\mathfrak{sl}_2(\mathbb{C})$-module homomorphism.

   (b) If $C : V_d \to V_d$, then by Schur's Lemma, $C(v) = cv$ for some $c \in \mathbb{C}$. Show that $c = d(d+2)/2$.