Math 2001: Homework 7

Due: October 22, 2008

Give complete justifications for all your answers.

Problem 1

1. Note that
$$20332 = 2 \cdot 299 \cdot 2 \cdot 17 = 391 \cdot 2 \cdot 13 \cdot 2.$$
Why does this not contradict the unique factorization of numbers into primes?

2. Write addition and multiplication tables for $\mathbb{Z}_3$ and $\mathbb{Z}_4$.

3. Suppose we wanted to "extend" the concept of divisibility to all integers, including 0. Let us say that an integer n is divisible by a number m if there exists an integer k such that $n = km$.
   (a) What numbers are divisible by 0?
   (b) What does $\mathbb{Z}_0$ look like? How many elements does it have, and what do the congruence classes look like?
   (c) What does $\mathbb{Z}_{-n}$ look like for negative integers $-n$?

Problem 2

1. Which of the following "rules" are true? Either prove or provide a counter-example.
   (a) If $a \equiv b \pmod{c}$, then $a + x \equiv b + x \pmod{c + x}$.
   (b) If $a \equiv b \pmod{c}$, then $ax \equiv bx \pmod{cx}$.

2. Prove that if p is prime, and $0 < k < p$, then p divides $\binom{p}{k}$. Why is p prime important for this to be true?
Hint: Use the factorial description of $\binom{p}{k}$.

3. Use the Binomial Theorem and the previous problem to show that if p is prime, then
$$(X + Y)^p \equiv X^p + Y^p \pmod{p}.$$