Worksheet 6: Rings

A *ring* $(R, +, \cdot)$ is a set R with two functions

$$+ : \begin{array}{ccc} R \times R & \longrightarrow & R \\ (r, s) & \mapsto & r + s \end{array} \quad \text{and} \quad \cdot : \begin{array}{ccc} R \times R & \longrightarrow & R \\ (r, s) & \mapsto & rs \end{array}$$

such that

(R1) $(R, +)$ is an abelian group,

(R2) for $r, s, t \in R$,

$$(r + s)t = rt + st, \quad r(s + t) = rs + rt, \quad r(st) = (rs)t.$$

(R3) there exists $1 \in R$ such that $r \cdot 1 = r = 1 \cdot r$ for all $r \in R$.

Questions:

1. Show that $(\mathbb{R}, +, \cdot)$, $(\mathbb{Z}, +, \cdot)$ and $(\mathbb{Z}_n, +_n, \cdot_n)$ are rings.
2. Show that $(M_n(R), +, \text{matrix multiplication})$ is a non-commutative ring.
3. If R is a ring, let

   $$R^\times = \{r \in R \mid \text{ there exists } r^{-1} \in R \text{ such that } r^{-1} \cdot r = 1 = r \cdot r^{-1}\}.$$

   Find $R^\times$ for the rings in Problem 1.
4. Show that $R^\times$ is a group.
5. What is a precise definition for *subring* and for *ring isomorphism*?