Worksheet 3: Symmetric groups

Recall, that a symmetry is a bijection on a set that respects some structure that we care about. The symmetric groups are the "largest" group of symmetries for sets. Specifically, for $n \in \mathbb{Z}_{\geqslant 1}$, let

$$S_n = \{w : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\} \mid w \text{ is a bijection}\}.$$

We typically refer to the elements of S_n as *permutations*.

1. What is the order of S_n?

2. Is S_n abelian?

Given an element $w \in S_n$, we can encode the information of this function into cycles, giving its *cycle decomposition*. For example,

$$\begin{array}{rcl} 1 & \mapsto & 3 \\ 2 & \mapsto & 12 \\ 3 & \mapsto & 8 \\ 4 & \mapsto & 2 \\ 5 & \mapsto & 5 \\ 6 & \mapsto & 10 \\ 7 & \mapsto & 9 \\ 8 & \mapsto & 1 \\ 9 & \mapsto & 7 \\ 10 & \mapsto & 6 \\ 11 & \mapsto & 4 \\ 12 & \mapsto & 11 \end{array} \quad \text{has cycle decomposition} \quad (1, 3, 8)\,(2, 12, 11, 4)\,(5)\,(6, 10)\,(7, 9)\,.$$

3. Compute the product $(1, 5, 4)(2, 3) \circ (1)(2, 5)(3, 4)$.

4. What is the order of a permutation $w \in S_n$ is based on its cycle decomposition?

5. How do you find the inverse of a permutation in cycle decomposition?

6. Find an explicit copy of D_n sitting inside S_n (ie. what should r and s be?).

A *cycle* in $w \in S_n$ is a sequence $(i_1, \dots, i_\ell)$ such that

$$w(i_1) = i_2,\ w(i_2) = i_3, \dots, w(i_\ell) = i_1.$$

We call the integer ℓ the *length* of the cycle. A *transposition* $w \in S_n$ is a permutation with all cycles of length 1 except one unique cycle of length 2 (we usually omit the 1-cycles from the cycle decomposition).

7. What is the order of a transposition?

8. Show that every permutation can be written as a product of transpositions.

Group assignment, due September 20, 2019

This assignment finds D_n as a subgroup of S_n. Give a description of how this works. You should include

1. Explicit permutations in S_n that realize $r, s \in D_n$, including their cycle decompositions.

2. Find a second different copy of D_n in S_n. What is the difference in terms of symmetries of a polygon implied by your choice?

3. Compute the intersection between the two copies you found.