Worksheet 10: Group actions and orbits

A *left action* of a group G on a set A is a function

$$\begin{array}{ccc} G \times A & \longrightarrow & A \\ (g, a) & \longmapsto & g(a) \end{array}$$

such that

(A1) $1(a) = a$ for all $a \in A$,

(A2) $g(h(a)) = (gh)(a)$ for all $g, h \in G$, $a \in A$.

In this case, we say G *acts* on A.

1. Define a left action of S_n on the power set $\mathcal{P}(\{1, 2, \ldots, n\})$.
2. Let A be a set. If $\varphi : G \to S_A$ is a homomorphism, define an action of G on A.
3. Prove that if G acts on A, then

$$\begin{array}{ccc} G & \longrightarrow & S_A \\ g & \longmapsto & w_g : \begin{array}{ccc} A & \to & A \\ a & \mapsto & g(a) \end{array} \end{array}$$

   is a homomorphism (don't forget to check that w_g is a bijection).

If G acts on A, then the *orbit* $G(a)$ containing $a \in A$ is the set

$$G(a) = \{g(a) \mid g \in G\}.$$

Note that the orbits of an action partition A.

4. What are the orbits in $\mathcal{P}(\{1, 2, \ldots, n\})$ of the action defined in 1.
5. Note that $D_n \subseteq S_n$ also acts on $\mathcal{P}(\{1, 2, \ldots, n\})$. What is the orbit $D_n(\{1, 2\})$? What are the other orbits of subsets of size 2?

Individual write-up (due 11.15.19). Give a description of the actions of S_n and D_n on $\mathcal{P}(\{1, 2, \ldots, n\})$ (be sure to explicitly say which copy of D_n you are using in S_n). Describe the orbits of the S_n-action in general, and the D_n-orbits of sets of size 2.