Math 3140: Homework 6

A. 9.1 Do either of the following sets of $n \times n$ matrices form a group under matrix multiplication?

(a) Diagonal matrices, $\{a \in M_n(\mathbb{R}) \mid a_{ij} = 0, i \neq j, a_{ii} \neq 0\}$.

(b) Symmetric matrices, $\{a \in M_n(\mathbb{R}) \mid a_{ij} = a_{ji}, 1 \leq i, j \leq n\}$.

9.3 Prove that the elements of $\mathrm{GL}_n(\mathbb{R})$ which have integer entries and determinant in $\{1, -1\}$ form a subgroup of $\mathrm{GL}_n(\mathbb{R})$. Note that this group is in fact $\mathrm{GL}_n(\mathbb{Z})$. Hint: Recall Cramer's Rule from linear algebra.

10.1 Show that if $G \times H$ is cyclic, then both G and H are cyclic.

10.2 Show that $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}$ are not isomorphic.

10.7 Which of the following groups are isomorphic to one-another?

$$\mathbb{Z}_{24}, \quad D_4 \times \mathbb{Z}_3, \quad D_{12}, \quad A_4 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times D_6, \quad S_4, \quad \mathbb{Z}_{12} \times \mathbb{Z}_2.$$

B. For p prime, let $\mathbb{F}_p$ denote the set $\{0, 1, \ldots, p-1\}$ where we add **and** multiply modulo p (as opposed to $\mathbb{Z}_p$ where we just add). Define

$$U_n(\mathbb{F}_p) = \{a \in M_n(\mathbb{F}_p) \mid a_{jj} = 1, 1 \leq j \leq n, a_{ji} = 0, 1 \leq i < j \leq n\}.$$

This group is called the *group of unipotent, uppertriangular matrices with entries in* $\mathbb{F}_p$.

(a) What is the order of $U_3(\mathbb{F}_2)$? Show that $U_3(\mathbb{F}_2)$ is isomorphic to an already familiar group.
Remark. The group $U_3(\mathbb{F}_p)$ is often called the *Heisenberg group* and is useful in mathematical physics.

(b) Show that $U_2(\mathbb{F}_p) \cong \mathbb{Z}_p$, and that if $n \geq 2$, then $U_2(\mathbb{F}_p)$ is isomorphic to a subgroup of $U_n(\mathbb{F}_p)$.