Math 3140: Homework 2

A. 3.1 Show that each of the following collections of numbers forms a group under addition.

   (i) The even integers.

   (ii) All real numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$.

   (iii) All real numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$.

   3.5 Let n be a positive integer. *Multiplication modulo* n is the function

$$\begin{array}{rccc} \cdot_n : & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \mathbb{Z}_n \\ & (a, b) & \mapsto & a \cdot_n b, \end{array}$$

   where $ab = mn + a \cdot_n b$ for some $m \in \mathbb{Z}$.

   Prove that

$$(x \cdot_n y) \cdot_n z = x \cdot_n (y \cdot_n z),$$

   for all $x, y, z \in \mathbb{Z}$.

   3.9 Let p be a prime number and let x be an integer which satisfies $1 \leq x \leq p - 1$. Show that none of $x, 2x, \ldots, (p-1)x$ is a multiple of p. Deduce the existence of an integer z such that $1 \leq z \leq p - 1$ and $x \cdot_p z = 1$.

   3.10 Use the results of 3.5 and 3.9 to verify that multiplication modulo n makes $\{1, 2, \ldots, n-1\}$ a group if n is prime.

B. 4.1, 4.4, 4.5, 4.6, 4.8