Math 3140: Homework 10

A.

15.10. Let $H \subseteq G$ be a subgroup, and suppose $H = \langle X \rangle$ for some subset $X \subseteq H$ and $G = \langle Y \rangle$ for some subset $Y \subseteq G$. Show that $H \triangleleft G$ if $yxy^{-1} \in H$ for all $x \in X$ and $y \in Y$.

15.14. i. Show that every element in $\mathbb{Q}/\mathbb{Z}$ has finite order.

ii. Show that the only element in $\mathbb{R}/\mathbb{Q}$ that has finite order is the identity.

B.

16.4. Let $A \triangleleft G$ and $B \triangleleft H$. Show that $A \times B \triangleleft G \times H$ and

$$(G \times H)/(A \times B) \cong (G/A) \times (H/B).$$

(see how heavily you can use the isomorphism theorems to prove this).

16.11. (Fourth Isomorphism Theorem) Let $\varphi : G \to H$ be a surjective homomorphism with kernel K.

(a) For every subgroup $B \subseteq H$, show that the set

$$\varphi^{-1}(B) = \{g \in G \ \mid \ \varphi(g) \in B\}$$

is a subgroup of G that contains K.

(b) Show that there is a bijection between

$$\left\{ \begin{array}{c} \text{Subgroups } A \subseteq G \\ \text{such that } K \subseteq A \end{array} \right\} \longleftrightarrow \left\{ \text{ Subgroups } B \subseteq H \ \right\}.$$

16.12 An *maximal* normal subgroup N of G is a normal subgroup such that if $H \supseteq N$ is a normal subgroup of G, then $H = N$ or $H = G$. Show that N is a maximal normal subgroup of G if and only if G/N is simple.