Research Statement

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Abstract

My research lies at the intersection of number theory and random matrix theory. I am interested in the connection between heights of polynomials and ensembles of asymmetric random matrices. While there are known links between number theory and random matrix theory (*e.g.* statistical similarities between the statistics of the zeros of the Riemann zeta functions and the eigenvalues of certain ensembles of random matrices), my research demonstrates a new connection between the two fields.

1 Number Theory and Volumes of Polynomials

Given a polynomial in $\mathbb{C}[x]$, its Mahler measure is the absolute value of the leading coefficient times the product of the absolute values of its roots outside the unit circle. Mahler measure is a height (measure of complexity) of polynomials, which for integer polynomials can be interpreted as 'cyclotomicness': the Mahler measure of an irreducible integer polynomial is 1 if and only if the polynomial is cyclotomic. It is still unknown if 1 is a limit point of Mahler measures of integer polynomials; this is Lehmer's Problem [11]. There are other open questions regarding the range of Mahler measure restricted to integer polynomials [1].

1.1 Counting Reciprocal Polynomials with Bounded Mahler Measure

A reciprocal polynomial is one whose coefficient vector is invariant under reversal of order. In 1971, Smyth proved that if the Mahler measure of an irreducible integer polynomial is less than $1.3\ldots$, then the polynomial is necessarily reciprocal ($x - 1$ is the sole exception). It is known that for every positive integer N and $T > 0$, there are a finite number of reciprocal integer polynomials of degree at most N and Mahler measure at most T. We will call this number $\mathcal{M}_N(T)$. In my thesis, I use geometry of numbers techniques to give asymptotic estimates for $\mathcal{M}_N(T)$ by computing the *volume* of coefficient vectors of reciprocal *real* polynomials of degree at most N and Mahler measure at most T [18]. We will call this number $\mathcal{V}_N(T)$. It turns out that $\mathcal{V}_N(T) = T^{N+1}\mathcal{V}_N(1)$. Thus if we regard Mahler measure as a distance function (in the sense of the geometry of numbers) on the coefficient vectors of degree N polynomials, the asymptotic estimates for $\mathcal{M}_N(T)$ reduce to the computation of $\mathcal{V}_N(1)$ — the volume of the unit *star body* (ball).

I proved that $\mathcal{V}_N(1)$ is an explicitly computable rational number and moreover that this number is most easily described as the product of N simpler rational numbers. This follows results of S.-J. Chern and J. Vaaler which show that the analogous volume of (not necessarily reciprocal) polynomials is also a rational number most easily expressed as a product [2]. On first inspection, these product formulations seem like a mere curiosity; however, they turn out to have implications in random matrix theory. My method for computing $\mathcal{V}_N(T)$ and producing estimates for $\mathcal{M}_N(T)$

drastically simplifies Chern and Vaaler's calculations and generalizes to a large class of multiplicative heights. These generalizations of Mahler measure were the primary focus of my thesis [18]. Applications of this theory to number theory are recorded in [19]. A similar analysis may also be carried out for volumes of complex polynomials; I have done this for complex reciprocal polynomials in my paper [17].

1.2 Conjugate Reciprocal Polynomials and the Link with Random Matrix Theory

A conjugate reciprocal polynomial is one whose coefficient vector is invariant under reversal followed by complex conjugation. David Farmer asked me if I could compute the volume of degree N monic *conjugate* reciprocal polynomials with all roots on the unit circle (call this set $\mathcal{W}_N$). David and his co-authors (F. Mezzadri and N. Snaith) were interested in zeros of conjugate reciprocal polynomials as models for zeros on the critical line of certain L-functions [7].

In a joint paper with Kathleen Petersen (Queen's University), I show that the volume of $\mathcal{W}_N$ is equal to the volume of the $(N-1)$-ball of radius 2 [13]. This is done by reducing the volume calculation to an integral done by Dyson in one of his first papers on random matrix theory [4]. We also show that $\mathcal{W}_N$ is homeomorphic to an $(N-1)$-ball and that the group of isometries is isomorphic to the dihedral group of order $2N$.

In a recent manuscript with Jeff Vaaler (University of Texas at Austin), we provide sufficient conditions for a degree N conjugate reciprocal polynomial to have all roots on the unit circle [21]. These results rely on the identification of special subsets of $\mathcal{W}_N$ which are defined by geometric conditions. This follows work of Schinzel [16], and Lakatos and Losonczi [10].

2 Ensembles of Asymmetric Random Matrices

In 1965 Ginibre introduced three ensembles of random matrices whose entries are chosen independently with Gaussian density from $\mathbb{R}$, $\mathbb{C}$ and Hamilton's Quaternions [8]. (In the language of random matrix theory an *ensemble* is a set of matrices together with a probability measure). The study of the eigenvalue statistics of Ginibre's real ensemble (GinOE) is difficult because the eigenvalues come in two flavors: real and complex conjugate pairs. Quantities of interest often fracture into unwieldy sums over all possible numbers of real and complex conjugate pairs of eigenvalues. Due to this complication, there are many basic questions surrounding GinOE which remain open.

2.1 Averages over Ginibre's Real Ensemble

Ensemble averages are key to the calculation of many quantities associated to GinOE. Connecting heights of polynomials with random matrix theory, I proved that $\mathcal{V}_N(1)$ can be written as the average of a function over GinOE. This is important since (as I mentioned in Section 1.1), $\mathcal{V}_N(1)$ is most easily expressed as a product. The method which produces this simple product formulation also allows for a product formulation for more general ensemble averages over GinOE [20]. In addition, this product formulation is (seemingly) independent of the decomposition of the space of eigenvalues. This allows us to write ensemble averages over GinOE using the exact same mechanism used for writing averages over the 'classical' ensembles of Dyson and Wigner.

Establishing this connection is significant since the correlation functions (and other basic quantities) for the classical ensembles can be computed from the averages of certain functions over these ensembles (see [22]). There is hope that this discovery will allow for analogous results in the case of GinOE.

3 Works in Progress and Proposed Future Directions

3.1 Current Projects

3.1.1 Correlation Functions for GinOE

(*Joint work with Francesco Mezzadri, University of Bristol*) Correlation functions give the probability that a random matrix has an eigenvalue in a neighborhood of each of a fixed number of points in the complex plane. Correlation functions are the key to more nuanced information about the spectrum of random matrices. A fundamental and important research direction is to see if the averages over GinOE can produce a closed form for the correlation functions for GinOE.

3.1.2 Universality for Asymmetric Real Matrices

(*Joint work with Francesco Mezzadri, University of Bristol*) The Universality Conjecture says (loosely) that as $N \to \infty$, the eigenvalue statistics of a random $N \times N$ matrices should depend not on the probability measure used to choose the matrix, but rather on the types of the entries (real, complex, etc.) and on the symmetries placed on the matrix (Hermitian, etc.). We wish to show that the large N eigenvalue statistics of real asymmetric matrices satisfy the Universality Conjecture for a wide class of probability measures.

3.1.3 Patterns and Periodicity in a Family of Resultants

(*Joint work with Kevin Hare and David McKinnon, University of Waterloo*) Given an irreducible monic polynomial $f(x) \in \mathbb{Z}[x]$ and a positive integer m, we may create a new monic polynomial $f_m(x) \in \mathbb{Z}[x]$ by raising the roots of f to the power m. In [3] E. Dobrowolski gives the best known lower bound for the Mahler measure of an integer polynomial based on the degree by using the fact that when p is prime, $p^{\deg f}$ divides the resultant of f and f_p. In [9] we generalize this fact to show if $m < n$ then $p^{(m+1)\deg f}$ divides $\mathrm{Res}(f_{p^n}, f_{p^m})$. Moreover, we prove that the function $(f, k) \mapsto \mathrm{Res}(f_j, f_k)$ is periodic modulo prime powers and provide an algorithm for computing the period based on the polynomial and the prime power.

3.2 Anticipated Future Projects

3.2.1 Limit Laws for the Spectral Radius as $N \to \infty$

Brian Rider at the University of Colorado at Boulder made the interesting observation that my theorem for the averages over GinOE should allow for the computation of the limit distribution for the spectral radius of matrices in GinOE as $N \to \infty$. I intend to investigate this problem further. Such a result would complement Brian's work for the limit distribution of the spectral radius for Ginibre's other ensembles in [14].

3.2.2 Density-Density Correlation Functions

Eugene Kanzieper at the Holon Institute of Technology observed that the density-density correlation functions can be computed via the average over GinOE of an appropriate function together with the so-called replica method. There are technical hurdles to determining such correlation functions, but assuming they can be surmounted, this would generalize a result of A. Edelman *et al.* which gives the mean density of eigenvalues of elements in GinOE [5][6].

3.2.3 Counting Algebraic Numbers with Bounded Height Whose Conjugates Satisfy Symmetries

Recently, David Masser and Jeff Vaaler have developed asymptotic estimates for the number of algebraic numbers of fixed N degree over a fixed number field K as the height of the algebraic numbers increases without bound [12]. This work is a generalization of results of Schanuel and others [15]. In their formulation, Masser and Vaaler allow for some flexibility in the choice of height. Replacing Mahler measure with one of my generalized heights should allow for analogous asymptotic estimates of algebraic numbers whose conjugates satisfy prescribed geometric symmetries.

3.3 Undergraduate Research Problems

Certain components of my research are accessible to the motivated undergraduate. For example, there are many valuable computational experiments that would shed light onto problems in random matrix theory and heights of polynomials. Moreover, many of the volume calculations in which I am interested lie in the domain of the geometry of numbers. As such, they naturally lend themselves to visualization. Although the analysis necessary for the volume calculations is rather intensive, the geometric picture could inspire the interest of undergraduates to pursue further mathematical research.

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