MATH 2300 (Calculus 2)

Spring 2007

2nd midterm exam — SOLUTIONS

1. **[10 pts]** Solve $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x$ by the method of integrating factors.

SOLUTION From the general form of the equation $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$ we obtain $p(x) = 2x$ and $q(x) = x$. Therefore

$$\mu(x) = \exp\left(\int p(x)\,\mathrm{d}x\right) = \exp\left(\int 2x\,\mathrm{d}x\right) = e^{x^2}$$

and

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x \quad &\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2} y(x)\right) = e^{x^2} x \\
&\Longrightarrow \quad \int \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2} y(x)\right)\right] \mathrm{d}x = \int e^{x^2} x\,\mathrm{d}x \\
&\Longrightarrow \quad e^{x^2} y(x) = C + \frac{1}{2} e^{x^2} \\
&\Longrightarrow \quad \boxed{y(x) = Ce^{-x^2} + \frac{1}{2}}
\end{aligned}$$

2. **[10 pts]** Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(1 + y^2\right)x$ by the method of separation of variables.

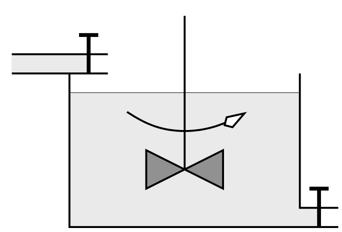
SOLUTION

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(1 + y^2\right)x \quad &\Longrightarrow \quad \frac{\mathrm{d}y}{1 + y^2} = 2x\,\mathrm{d}x \\
&\Longrightarrow \quad \int \frac{\mathrm{d}y}{1 + y^2} = \int 2x\,\mathrm{d}x \\
&\Longrightarrow \quad \arctan y = x^2 + C \\
&\Longrightarrow \quad \boxed{y = \tan\left(x^2 + C\right)}
\end{aligned}$$

3. **[15 pts]** At time $t = 0$ a tank contains 10 oz of salt dissolved in 20 gal of water. Then brine containing one oz of salt per gallon of brine is allowed to enter the tank at a rate of two gallons per minute, and the mixed solution is drained from the tank at the same rate. How much salt is in the tank at an arbitrary time t?

SOLUTION Let S be the amount of salt in the tank measured in ounces, then

$$\frac{\mathrm{d}S}{\mathrm{d}t} = [\text{rate salt}]_{in} - [\text{rate salt}]_{out}\,, \qquad S(0) = 10 \text{ oz}$$



with

$$\begin{aligned}
[\text{rate salt}]_{in} &= [\text{rate brine}]_{in} \times [\text{salt concentration in brine}] \\
[\text{rate salt}]_{out} &= [\text{rate fluid}]_{in} \times [\text{salt concentration fluid}]
\end{aligned}$$

Therefore

$$\begin{aligned}
[\text{rate salt}]_{in} &= 2\frac{\text{gal}}{\text{min}} \times 1\frac{\text{oz}}{\text{gal}} = 2\frac{\text{oz}}{\text{min}} \\
[\text{rate salt}]_{out} &= 2\frac{\text{gal}}{\text{min}} \times \frac{S \text{ oz}}{20 \text{ gal}} = \frac{S}{10}\frac{\text{oz}}{\text{min}}
\end{aligned}$$

The differential equation that models the problem is then

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 2 - \frac{S}{10}, \qquad S(0) = 10 \qquad\qquad \text{with } [S] = \text{oz} \quad \text{and} \quad [t] = \text{min.}$$

This differential equation is solved as follows

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} + \left(\frac{1}{10}\right) S = 2 \;&\Longrightarrow\; \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t/10} S\right) = 2e^{t/10} \\
&\Longrightarrow\; \int \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t/10} S\right)\right] \mathrm{d}t = \int 2e^{t/10}\, \mathrm{d}t \\
&\Longrightarrow\; e^{t/10} S = C + 20e^{t/10} \\
&\Longrightarrow\; S(t) = Ce^{-t/10} + 20
\end{aligned}$$

The initial condition will fix the value of the integration constant C

$$S(0) = C + 20 = 10 \qquad \Longrightarrow \qquad C = -10$$

so, finally

$$\boxed{S(t) = 20 - 10e^{-t/10} \quad \text{oz}, \quad t \text{ in min.}}$$

4. **[10 pts]** Find the general solution of the differential equation $y'' - 6y' + 9y = 0$.

SOLUTION The characteristic equation of this *second order linear homogeneous* differential equation is

$$m^2 - 6m + 9 = 0 \qquad \Longrightarrow \qquad m = \frac{6 \pm \sqrt{36 - 36}}{2} = 3 \quad \text{(double root!)}$$

The two independent solutions of the homogeneous equation are then

$$y_1(x) = e^{3x} \qquad \text{and} \qquad y_2(x) = x\, e^{3x}$$

therefore,

$$\boxed{y_{\mathrm{GS}}(x) = c_1 e^{3x} + c_2 x\, e^{3x} = (c_1 + c_2 x)\, e^{3x}}$$

5. **[15 pts]** Find the limit, if they exist, of the following sequences

(a) $\left\{\ln\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$

(b) $\left\{\frac{\pi^n}{4^n}\right\}_{n=1}^{\infty}$

(c) $\left\{\frac{\cos^2 n}{n}\right\}_{n=1}^{\infty}$

SOLUTION

(a)

$$\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = \lim_{x\to 0} \ln x = -\infty$$

(b)

$$\lim_{n\to\infty} \frac{\pi^n}{4^n} = \lim_{n\to\infty} \left(\frac{\pi}{4}\right)^n = 0 \qquad \left(\frac{\pi}{4} < 1\right)$$

(c)

$$0 \le \frac{\cos^2 n}{n} \le \frac{1}{n} \qquad \Longrightarrow \qquad \lim_{n\to\infty} \frac{\cos^2 n}{n} = 0$$

6. **[10 pts]** Consider the sequence

$$a_1 = 1, \qquad a_2 = 1 + \frac{1}{2}, \qquad a_3 = 1 + \cfrac{1}{2 + \cfrac{1}{2}}, \qquad a_4 = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}, \qquad a_5 = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}, \qquad \cdots$$

Find a recursion formula for a_{n+1} and the limit of the sequence assuming that such a limit exists.

SOLUTION The recursion relation for the sequence is

$$\boxed{a_1 = 1, \qquad a_{n+1} = 1 + \frac{1}{1 + a_n}}$$

If $\lim_{n\to\infty} a_n = L$, then

$$L = 1 + \frac{1}{1+L} \quad \Rightarrow \quad L - 1 = \frac{1}{L+1} \quad \Rightarrow \quad (L-1)(L+1) = 1 \quad \Rightarrow \quad L^2 = 2 \quad \Rightarrow \quad \boxed{L = \sqrt{2}}$$

7. **[10 pts]** Find out whether the following series are eventually strictly increasing or strictly decreasing.

(a) $\left\{\dfrac{10^n}{n!}\right\}_{n=1}^{\infty}$

(b) $\{ne^{2n}\}_{n=1}^{\infty}$

SOLUTION

(a) Since there are factorials involved in the series, the ratio a_{n+1}/a_n is the most convenient approach.

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{10^{n+1}}{(n+1)!}}{\dfrac{10^n}{n!}} = \frac{10^{n+1}}{10^n}\,\frac{n!}{(n+1)!} = \frac{10 \cdot 10^n}{10^n}\,\frac{n!}{(n+1)\cdot n!} = \frac{10}{(n+1)}$$

$$a_n > 0, \quad \forall\, n, \quad \text{and} \quad \frac{a_{n+1}}{a_n} < 1, \quad \forall\, n \geq 10 \qquad \Longrightarrow \qquad \left\{\frac{10^n}{n!}\right\}_{n=1}^{\infty} \quad \text{eventually } \textbf{decreasing}$$

(b) Consider $f(x) = xe^{2x}$. Then $a_n = f(n)$, and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^{2x}\right] = (1+2x)\,e^{2x} > 0, \quad \forall\, x > 0 \qquad \Longrightarrow \qquad \{ne^{2n}\}_{n=1}^{\infty} \quad \text{eventually } \textbf{increasing}$$

8. **[10 pts]** Determine whether the series converges, and if so find its sum.

(a) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k+2}$

(b) $\displaystyle\sum_{k=1}^{\infty} 5^{3k} 7^{1-k}$

SOLUTION

(a) Writing the given sum in terms of a *geometric series* we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k+2} &= \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^6 + \cdots \\
&= \left(\frac{2}{3}\right)^3 \left[1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right] \\
&= \left(\frac{2}{3}\right)^3 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \left(\frac{2}{3}\right)^3 \frac{1}{1 - \dfrac{2}{3}} = \left(\frac{2}{3}\right)^3 3 = \frac{8}{9}.
\end{aligned}$$

(b) Writing the given sum in terms of a *geometric series* we have

$$\sum_{k=1}^{\infty} 5^{3k} \cdot 7^{1-k} = \sum_{k=1}^{\infty} 5^{3k} \cdot 7 \cdot 7^{-k} = 7 \sum_{k=1}^{\infty} 5^{3k} \cdot 7^{-k} = 7 \sum_{k=1}^{\infty} \left(5^3\right)^k \left(\frac{1}{7}\right)^k = 7 \sum_{k=1}^{\infty} \left(\frac{5^3}{7}\right)^k = \infty$$

because the ratio of the series is $\frac{5^3}{7} > 1$. (There is no need of wasting your time starting the sum from 0.)

9. **[10 pts]** Determine whether the series converges, and if so find its sum.

(a) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right)$

(b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k-2}$

SOLUTION

(a) The sum $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right)$ is a *telescoping* sum. Its *partial sum* s_n is

$$\begin{aligned} s_n &= \left(\frac{1}{2^1} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \left(\frac{1}{2^3} - \frac{1}{2^4}\right) + \cdots + \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right) \\ &= \frac{1}{2} + \left(-\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(-\frac{1}{2^3} + \frac{1}{2^3}\right) + \left(-\frac{1}{2^4} + \frac{1}{2^4}\right) + \cdots - \frac{1}{2^{n+1}} \\ &= \frac{1}{2} - \frac{1}{2^{n+1}} \end{aligned}$$

therefore

$$\sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = \lim_{n\to\infty} s_n = \frac{1}{2} \cdot$$

(b) Expanding the series $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k-2}$ we have

$$\sum_{k=3}^{\infty} \frac{1}{k-2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k}$$

which is the *harmonic series*. Therefore $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k-2} = \infty$.