MATH 2300: CALCULUS 2

NOVEMBER 15, 2006

MIDTERM 3

I have neither given nor received aid on this exam.

Name:____

○ **001** J. Newhall .... (9am)

○ **002** S. Preston .... (10am)

○ **003** K. Kearnes .... (11am)

○ **004** S. Preston .... (12pm)

○ **005** J. Wiscons .... (2pm)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

A nice formula:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	30 pts	
2	30 pts	
3	30 pts	
4	30 pts	
5	30 pts	
6	30 pts	
7	30 pts	
8	30 pts	
TOTAL	240 pts	

1. True or False? Justify your answer with a brief explanation or a counterexample.

(a) **(10 points)** A geometric series cannot be conditionally convergent.

True. $\sum_{k=0}^{\infty} ar^k$ converges absolutely for $|r| < 1$ and diverges if $|r| \geq 1$, so there is no value of r for which it converges conditionally.

(b) **(10 points)** $\sum_{k=1}^{\infty} (x^2 - 1)^k$ is a power series.

False. A power series has the form $\sum_{k=1}^{\infty} a_k (x - c)^k$, with each term a product of constant a_k and a power of the same linear term $(x - c)$, and this series is not in this form.

(Remark: Another way to see that the given series is not a power series is that its convergence set is not an interval, it is $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$. The convergence set of any power series is an INTERVAL according to Theorem 10.8.3.)

(c) **(10 points)** If the Maclaurin series for $f(x)$ converges on $[-1, 1]$, then the Maclaurin series for $f'(x)$ also converges on $[-1, 1]$.

False. $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ converges on $[-1, 1]$, but its derivative $\sum_{k=1}^{\infty} \frac{x^{k-1}}{k}$ converges only on $[-1, 1)$.

2. **(30 points)** Show that if $\{a_k\}_{k=1}^{\infty}$ is monotone increasing, $\{b_k\}_{k=1}^{\infty}$ is monotone decreasing, and $a_k < b_k$ for all k, then $\lim\limits_{k\to\infty} a_k$ and $\lim\limits_{k\to\infty} b_k$ both exist.

The sequence $\{a_k\}_{k=1}^{\infty}$ is increasing and bounded above by b_1 (since $a_k < b_k \le b_1$), so by the Completeness Axiom it has a limit. Similarly, $\{b_k\}_{k=1}^{\infty}$ is decreasing and bounded below by a_1, so it has a limit.

3. Test the following series for convergence. (State which tests you are applying.)

(a) **(10 points)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[k]{k}}$

The series diverges by the n-th term test for divergence: $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, so $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{\sqrt[n]{n}} \neq 0$.

(b) **(10 points)** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

The series converges by the integral test: $\displaystyle\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} [\tan^{-1}(b) - \tan^{-1}(1)] = \pi/4.$

(c) **(10 points)** $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

The series converges by the root test:

$$\rho = \lim_{k\to\infty} (a_k)^{1/k} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \lim_{k\to\infty} \left(\frac{1}{(1+\frac{1}{k})^k}\right) = \frac{1}{e},$$

so $\rho < 1$.

4. For which values of $p \geq 0$ do the following series converge?

(a) **(15 points)** $\sum_{k=1}^{\infty} k^p p^k$.

By the root test we have $\rho = \lim_{k\to\infty} (k^p p^k)^{1/k} = (\lim_{k\to\infty} \sqrt[k]{k})^p p = p$, so the series converges if $\rho = p < 1$ and diverges if $\rho = p > 1$. If $p = 1$, then the series diverges by the n-th term test for divergence. Altogether, the series converges only when $0 \leq p < 1$.

(b) **(15 points)** $\sum_{k=1}^{\infty} \frac{e^{2pk}+1}{e^{pk}}$.

$\sum_{k=1}^{\infty} \frac{e^{2pk}+1}{e^{pk}}$ diverges for every $p \geq 0$. To see this, note that the k-th term is $e^{pk} + e^{-pk}$ and $e^{pk} + e^{-pk} > e^{pk} \geq e^0 = 1$, so the terms do not go to zero.

5. **(30 points)** Find the interval of convergence of $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k\ln(k)}$.

There is an error in this problem, the sum should start at $k = 2$ (otherwise it isn't defined).

For the corrected version, the ratio test yields

$$\lim_{k\to\infty} \left| \frac{\frac{(x-1)^{k+1}}{(k+1)\ln(k+1)}}{\frac{(x-1)^k}{k\ln(k)}} \right| = \lim_{k\to\infty} \left| (x-1) \frac{k\ln(k)}{(k+1)\ln(k+1)} \right| = |x-1|,$$

so the series converges if $|x-1| < 1$ and diverges if $|x-1| > 1$. Therefore the interval of convergence is centered at $c = 1$ and has radius $R = 1$. To test the endpoints, we substitute $x = 0$ to get $\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln(k)}$, which converges by the alternating series test, and substitute $x = 2$ to get $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$, which diverges by the integral test. This shows that the interval of convergence is $[0, 2)$.

6. Find the Maclaurin series for $f(x)$ by any method, and determine the radius of convergence.

(a) **(15 points)** $f(x) = e^{-x^2/2}$.

Substitute $-x^2/2$ into the Maclaurin series for e^x:

$$M(x) = 1 - \frac{x^2}{2 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \frac{x^8}{2^4 \cdot 4!} - \cdots .$$

The radius of convergence is ∞, since that is true for the Maclaurin series for e^x.

(b) **(15 points)** $f(x) = \dfrac{x^3}{1-2x}$.

Manipulate a geometric series:

$$\begin{aligned} M(x) = x^3 \left(\frac{1}{1-(2x)} \right) &= x^3 \left(1 + (2x) + (2x)^2 + (2x)^3 + \cdots \right) \\ &= x^3 + 2x^4 + 4x^5 + 8x^6 + \cdots , \end{aligned}$$

valid if $|2x| < 1$, or $|x| < 1/2$.

The radius of convergence is $1/2$.

7. (a) **(15 points)** Find the Taylor series, $T(x)$, for $f(x) = \sinh(x)$ centered at $c = 1$.

Using the facts that $[\sinh(x)]' = \cosh(x)$ and $[\cosh(x)]' = \sinh(x)$, we get

$$T(x) = \sinh(1) + \frac{\cosh(1)}{1!}(x-1) + \frac{\sinh(1)}{2!}(x-1)^2 + \frac{\cosh(1)}{3!}(x-1)^3 + \cdots .$$

(b) **(15 points)** Use a remainder term estimate to prove that $f\left(\frac{1}{2}\right) = T\left(\frac{1}{2}\right)$.

$|R_n(1/2)| \leq \dfrac{M}{(n+1)!}(1/2)^{n+1}$ where M is an upper bound on $|f^{(n+1)}(x)|$ on the interval $[1/2, 1]$. Since $f^{(n+1)}(x) = \sinh(x)$ or $\cosh(x)$, and $\sinh(x) \leq \cosh(x) \leq \cosh(1)$ on this interval, we can take $M = \cosh(1)$. Thus,

$$0 \leq \lim_{n\to\infty} |R_n(1/2)| \leq \lim_{n\to\infty} \cosh(1) \cdot \frac{(1/2)^{n+1}}{(n+1)!} = 0$$

by a basic limit from class $\left(\text{namely } \lim_{n\to\infty} \frac{a^{n+1}}{(n+1)!} = 0 \text{ holds for any } a\right)$.

(Remark: It is not necessary to compute a value for M, only to show that some M exists that works for all n. For this, it is enough to note that both $\sinh(x)$ and $\cosh(x)$ are continuous on $[1/2, 1]$, so each has a maximum on $[1/2, 1]$. If M is the larger of the two maximums, then since $f^{(n+1)}(x) = \sinh(x)$ or $\cosh(x)$ we have $|f^{(n+1)}(x)| \leq M$ on $[1/2, 1]$ for all n.)

8. (a) **(15 points)** Express $\int_0^1 \frac{\sin(x)}{x}dx$ as a series of real numbers.

$$\begin{aligned}\int_0^1 \frac{\sin(x)}{x}dx &= \int_0^1 \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x}dx \\ &= \int_0^1 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\ dx \\ &= \left[x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \cdots\right]_0^1 \\ &= 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \cdots \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{(2k-1)(2k-1)!}\end{aligned}$$

(Remark: Either one of the last two lines of this calculation is a complete answer.)

(b) **(15 points)** How many terms of your series should you sum in order to approximate this integral to two decimal places? (This means that your error should be ≤ 0.005.)

Since the series from (a) satisfies the hypotheses of the alternating series test, if we sum from a_1 to a_n, then the term a_{n+1} bounds the error. Hence we want n so that

$$|a_{n+1}| = \frac{1}{(2n+1)(2n+1)!} \leq .005.$$

Inverting, this means that $(2n+1)(2n+1)! \geq 200$. If $n = 1$, then $3 \cdot 3! = 18 \ngeq 200$, but if $n = 2$, then $5 \cdot 5! = 600 \geq 200$. Hence we only need to sum the first $n = 2$ terms.