MATH 2300: CALCULUS 2

September 20, 2006

MIDTERM 1

I have neither given nor received aid on this exam.

Name:____

- ◯ **001** J. Newhall . . . . (9am)
- ◯ **002** S. Preston . . . . (10am)
- ◯ **003** K. Kearnes . . . . (11am)
- ◯ **004** S. Preston . . . . (12pm)
- ◯ **005** J. Wiscons . . . . (2pm)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

Some useful formulas:

$$\int \sqrt{u^2+1}\,du = \tfrac{u}{2}\sqrt{u^2+1} + \tfrac{1}{2}\ln\left(u+\sqrt{u^2+1}\right) + C$$

$$\int \sqrt{u^2-1}\,du = \tfrac{u}{2}\sqrt{u^2-1} - \tfrac{1}{2}\ln\left(u+\sqrt{u^2-1}\right) + C$$

DO NOT WRITE IN THIS BOX!

Problem	**Points**	**Score**
1	30 pts	
2	30 pts	
3	30 pts	
4	30 pts	
5	30 pts	
6	30 pts	
7	30 pts	
8	30 pts	
TOTAL	240 pts	

1. (30 points) If $\int_0^1 f(x)\,dx = 12$, then what is $\int_0^1 xf(x^2)\,dx$?

`If` $u = x^2$, `then` $du = 2x\,dx$, `so`

$$\int_0^1 xf(x^2)\,dx = \frac{1}{2}\int_0^1 f(\underline{x^2})\,\underline{2x\,dx} = \frac{1}{2}\int_0^1 f(u)\,du = \frac{1}{2}(12) = 6.$$

2. (30 points) Starting with the integral definition of the logarithm function, which is

(i) $\ln(x) = \int_1^x \frac{dt}{t}$,

related concepts are defined as follows.

(ii) $\exp(x)$ = the inverse of the function in (i).

(iii) $a^x = \exp(x\ln(a))$.

(iv) $\log_a(x)$ = the inverse of the function in (iii).

Use **these definitions** to show that the change of base formula, $\log_a(x) = \frac{\ln(x)}{\ln(a)}$, is valid. $\Big($Hint: start by writing $y = \log_a(x)$, then use these definitions to show that $y = \frac{\ln(x)}{\ln(a)}$.$\Big)$

`Let` $y = \log_a(x)$. `By (iv), this is equivalent to` $a^y = x$. `By (iii),` $\exp(y\ln(a)) = x$. `By (ii), this is equivalent to` $y\ln(a) = \ln(x)$.
`Hence` $y = \frac{\ln(x)}{\ln(a)}$.

3. Evaluate the following indefinite integrals.

(a) **(10 points)** $\int \theta \tan(\theta^2)\, d\theta$

The substitution $u = \theta^2$ $(du = 2\theta\, d\theta)$ reduces this integral to $\frac{1}{2}\int \tan(u)\, du = \frac{1}{2}\int \frac{\sin(u)}{\cos(u)}\, du$. Now the substitution $v = \cos(u)$ $(dv = -\sin(u)\, du)$ reduces this integral to $-\frac{1}{2}\int \frac{dv}{v} = -\frac{1}{2}\ln|v| + C = -\frac{1}{2}\ln|\cos(u)| + C = -\frac{1}{2}\ln|\cos(\theta^2)| + C$.

(b) **(10 points)** $\int \frac{1}{t(1+\ln(t^2))}\, dt$

The substitution $u = (1+\ln(t^2))$ $(du = \frac{2}{t}\, dt)$ reduces this integral to $\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+\ln(t^2)| + C$.

(c) **(10 points)** $\int \ln\left(\sqrt[w]{w}\right)\, dw$

We will use the fact that $\ln\left(\sqrt[w]{w}\right) = \ln\left(w^{\frac{1}{w}}\right) = \frac{1}{w}\ln(w)$.
Now, the substitution $u = \ln(w)$ $(du = \frac{dw}{w})$ reduces this integral to $\int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}\ln^2(w) + C$.

4. **(30 points)** Parametrize $(x-1)^2 + (y+3)^2 = 1$ in a way that traverses this circle 3 times in the clockwise direction.

$x(t) = 1 + \cos(-t), y(t) = -3 + \sin(-t)$, $t \in [0, 6\pi]$.
(Other answers may be correct.)

5. (a) **(10 points)** Sketch the curve defined parametrically by $x(t) = e^{-t}\cos(t)$ and $y(t) = e^{-t}\sin(t)$.

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(A picture of an exponential spiral that decays
in the counterclockwise direction goes here.)
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(b) **(10 points)** Find the length of this curve over the time interval $[0, b]$.

`First,` $x'(t) = -e^{-t}\cos(t) - e^{-t}\sin(t)$ `and` $y'(t) = -e^{-t}\sin(t) + e^{-t}\cos(t)$`,` `so`

$$\begin{aligned}\int_0^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt &= \int_0^b \sqrt{2e^{-2t}(\cos^2(t) + \sin^2(t))}\,dt \\ &= \int_0^b \sqrt{2}e^{-t}\,dt \\ &= [-\sqrt{2}e^{-t}]_0^b \\ &= \sqrt{2}[1 - e^{-b}].\end{aligned}$$

(c) **(10 points)** What happens to the length as b goes to infinity?

$$\lim_{b\to\infty} \sqrt{2}[1 - e^{-b}] = \sqrt{2}[1 - \lim_{b\to\infty} e^{-b}] = \sqrt{2}.$$

6. **(30 points)** Find the length of the arc defined by $x(t) = \cos^3(t)$, $y(t) = \sin^3(t)$, $t \in [0, \pi/2]$.

`First,` $x'(t) = -3\cos^2(t)\sin(t)$ `and` $y'(t) = 3\sin^2(t)\cos(t)$`,` `so`

$$\begin{aligned}\int_0^{\pi/2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt &= \int_0^{\pi/2} \sqrt{9\sin^2(t)\cos^2(t)(\cos^2(t) + \sin^2(t))}\,dt \\ &= \int_0^{\pi/2} 3\sin(t)\cos(t)\,dt \\ &= 3[\tfrac{1}{2}\sin^2(t)]_0^{\pi/2} \\ &= 3/2.\end{aligned}$$

7. (30 points) Find the area of the surface obtained by revolving one arch of the cosine function around the x-axis.

One arch of the graph of $y = \cos(x)$ is the part of the curve with $x \in [-\pi/2, \pi/2]$.

$$\text{Area} = \int_{-\pi/2}^{\pi/2} 2\pi \cos(x) \sqrt{1 + [-\sin(x)]^2}\, dx.$$

The substitution $u = \sin(x)$ ($du = \cos(x)\, dx$) transforms this into $2\pi \int_{-1}^{1} \sqrt{1+u^2}\, du$. Using the formula from the front of the test we get

$$2\pi \int_{-1}^{1} \sqrt{1+u^2}\, du = 2\pi \left[\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2} \ln\left(u + \sqrt{u^2+1}\right)\right]_{-1}^{1} = 2\pi[\sqrt{2} + \ln(1+\sqrt{2})].$$

8. (a) **(15 points)** What is the average value of e^x on $[0, 10]$?

$$[e^x]_{\text{ave}} = \frac{1}{10} \int_0^{10} e^x\, dx = \frac{1}{10}[e^x]_0^{10} = (e^{10} - 1)/10.$$

(b) **(15 points)** Find a function whose average value on the interval $[0, x]$ is e^x for every x.

We want a function f such that

$$e^x = f_{\text{ave}} = \frac{1}{x} \int_0^x f(t)\, dt,$$

or

$$xe^x = \int_0^x f(t)\, dt.$$

Differentiating with respect to x yields $xe^x + e^x = f(x)$, or $f(x) = (x+1)e^x$.