1 The Integral Test

Let $\sum a_n$ be a positive-term series and f a positive-valued, decreasing, continuous function for $x \geq 1$. If $f(n) = a_n$ for all integers $n \geq 1$, then the series and improper integral

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_{1}^{\infty} f(x)dx$$

either both converge or both diverge.

1.1 Example

Show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

2 The Comparison Test

Let $\sum a_n$ and $\sum b_n$ be positive-term series. Suppose $a_n \leq b_n$ for all integers $n \geq 1$, then:

1. $\sum a_n$ converges if $\sum b_n$ converges
2. $\sum b_n$ diverges if $\sum a_n$ diverges

2.1 Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{e^n}$ converges.

3 The Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be positive-term series. If the limit $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists and $0 < L < +\infty$, then either both series converge or both series diverge.

3.1 Example

Determine whether the series $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right)$ converges.

4 The Alternating Series Test

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ satisfies:

a) $a_n \geq a_{n+1} > 0$ for all n (i.e., $\{a_n\}$ is monotonically decreasing.), and
b) $\lim_{n\to\infty} a_n = 0$

Then the alternating series converges.

4.1 Example

Determine whether the series is absolutely convergent, conditionally convergent, or divergent. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{1+n^2}$

5 The Ratio Test

Let $\sum a_n$ be a given series. If the limit $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists or is infinite, then:

1. $\sum a_n$ converges absolutely if $L < 1$
2. $\sum a_n$ diverges if $L > 1$

If $L = 1$, then the Ratio Test is inconclusive.

5.1 Example

Determine whether the series is absolutely convergent, conditionally convergent, or divergent. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n! n}{2^n}$

6 The Root Test

Let $\sum a_n$ be a given series. If the limit $L = \lim_{n\to\infty} (|a_n|)^{\frac{1}{n}}$ exists or is infinite, then:

1. $\sum a_n$ converges absolutely if $L < 1$
2. $\sum a_n$ diverges if $L > 1$

If $L = 1$, then the Root Test is inconclusive.

6.1 Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ converges.