Coxeter Group Actions on Complementary Pairs of Very Well-Poised ${}_9F_8(1)$ Hypergeometric Series

by

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Coxeter Group Actions on Complementary Pairs of Very Well-Poised ${}_9F_8(1)$ Hypergeometric Series

Thesis directed by Prof. Eric Stade

We investigate a function $J(\vec{x}) = J(a; b; c, d, e, f, g, h)$, which is a linear combination of two very well-poised ${}_9F_8(1)$ hypergeometric series. We first show that J is invariant under the action of a certain matrix group G_J isomorphic to the Coxeter group $W(E_6)$ of order 51840, acting on the affine hyperplane $V = \{(a, b, c, d, e, f, g, h)^T \in \mathbb{C}^8 \colon 2 + 3a = b + c + d + e + f + g + h\}$. We further develop an "algebra" of three-term relations for $J(\vec{x})$ and show that for any three elements μ_1, μ_2, μ_3 of a certain matrix group M_J isomorphic to the Coxeter group $W(E_7)$ of order 2903040 and containing the above group G_J, there is a relation among $J(\mu_1\vec{x})$, $J(\mu_2\vec{x})$, and $J(\mu_3\vec{x})$ in which the coefficients are rational combinations of gamma and sine functions in $\vec{x}$, provided that no two of the μ_j are in the same right coset of G_J in M_J.

This set of $\binom{|M_J|/|G_J|}{3} = \binom{56}{3} = 27720$ resulting three-term relations is then divided into five families based on the orbits of a certain group action and corresponding to the Euclidean type of the triple (μ_1, μ_2, μ_3). This Euclidean type is defined in terms of the Euclidean distances between vectors in a set corresponding to the elements of $G_J \backslash M_J$.

Each three-term relation of a given Euclidean type may be transformed into any other of the same type by a change of variable. We provide an explicit example of each of the five types of three-term relations, and show that the number of monomials of the coefficient of a given J function in a given three-term relation is related to the Euclidean distance between the other two J functions in that relation.

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Chapter 1

Introduction

1.1 Hypergeometric Series

Let $a \in \mathbb{C}$. We define the *Pochhammer symbol* $(a)_k$ by

$$(a)_0 = 1$$

and

$$(a)_k = a(a+1)(a+2)\dots(a+k-1) \text{ for } k \in \mathbb{Z}^+.$$

From the functional equation $\Gamma(1+s) = s\Gamma(s)$ for the gamma function, we see that for $a \neq 0, -1, -2, \dots,$ we have

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

The *hypergeometric series*

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}z^3 + \dots \\ &= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k \end{aligned} \tag{1.1.1}$$

was introduced by Gauss in 1821 [12], who studied many of its properties and demonstrated its relations to a great variety of elementary and special functions.

For positive integers p and q and $e_1, \dots, e_p, f_1, \dots f_q \in \mathbb{C}$, we define the *(generalized) hypergeometric series of type* ${}_pF_q$ to be

$${}_pF_q(e_1, \dots, e_p; f_1, \dots, f_q; z) = \sum_{k=0}^{\infty} \frac{(e_1)_k \dots (e_p)_k}{k!(f_1)_k \dots (f_q)_k} z^k.$$

We will also use

$$ {}_pF_q \left[\begin{matrix} e_1, \ldots, e_p \\ f_1, \ldots, f_q \end{matrix} ; z \right] $$

as an alternate notation for ${}_pF_q(e_1, \ldots, e_p; f_1, \ldots, f_q; z)$.

So that this series is well-defined, we assume that no f_j is equal to zero or a negative integer. If some e_j is zero or a negative integer, then $(e_j)_k$ will equal 0 for k sufficiently large, and so we say that the series *terminates*.

Generalized hypergeometric functions were introduced in the late 1800's and, from this time through the early 1900's, properties of and relations among generalized hypergeometric series were widely studied (for example, see [1], [2], [3], [30], [33], and [34]). Generalized hypergeometric series—and especially those of *unit argument*, meaning $z = 1$—became an object of renewed interest towards the end of the 20th century, due in part to their appearance in atomic and molecular physics (for example, see [7, Sections 2.7 and 2.9] and [15, Chapters 8, 9, and 11]) and in part to their presence in the theory of archimedean zeta integrals for automorphic L functions (for example, see [6], [25], [26], [27] [28], and [29]). It is now conventional to drop the adjective "generalized" and we will henceforth follow this convention.

Both "two-term" and "three-term" relations among hypergeometric series were early objects of interest of, for example, Thomae [30] and Whipple [33] [34]. In more recent research in the theory of hypergeometric series and of related entities such as "basic hypergeometric series," group-theoretic notions have been introduced to explain certain known relations among hypergeometric series of unit argument or among basic hypergeometric and other analogous series (for example, see [4], [5], [17], [18], and [20]). We will develop such a framework to describe our two- and three-term relations below.

A hypergeometric series of type ${}_pF_{p-1}$ converges absolutely if $|z| < 1$, or if $|z| = 1$ and

$$ \Re \left(\sum f_j - \sum e_j \right) > 0 $$

(see [1, Chapter 2]). If $1 + e_1 = f_1 + e_2 = \cdots = f_{p-1} + e_p$, then the series is said to be *well-poised*. If, in addition, $e_2 = 1 + \frac{e_1}{2}$, then the series is said to be *very well-poised*. When $z = 1$, the series is said to be of type ${}_pF_{p-1}(1)$. In this thesis, we will consider very well-poised ${}_9F_8(1)$ hypergeometric series.

Let V be the affine hyperplane

$$V = \{(a, b, c, d, e, f, g, h)^T \in \mathbb{C}^8 \colon 2 + 3a = b + c + d + e + f + g + h\}.$$

Given $(a, b, c, d, e, f, g, h)^T \in V$, consider the transformation

$$(a, b, c, d, e, f, g, h)^T \mapsto (2b - a, b, b + c - a, b + d - a, b + e - a, b + f - a, b + g - a, b + h - a)^T$$

of V, which amounts to an addition of $b - a$ to each vector coordinate, followed by a transposition of the first two coordinates.

We will call the two well-poised ${}_9F_8(1)$ hypergeometric series

$${}_9F_8\left[\begin{matrix} a, 1 + a/2, b, c, \ldots, h \\ a/2, 1 + a - b, 1 + a - c, \ldots, 1 + a - h \end{matrix}; 1\right]$$

and

$${}_9F_8\left[\begin{matrix} 2b - a, 1 - a/2 + b, b, b - a + c, b - a + d, \ldots, b - a + h \\ -a/2 + b, 1 + b - a, 1 + b - c, 1 + b - d, \ldots, 1 + b - h \end{matrix}; 1\right]$$

obtained from this transformation *complementary with respect to the parameter* b.

1.2 Barnes Integrals

Consider a function of the form

$$f(t) = \prod_{i=1}^{m} \Gamma^{\epsilon_i}(a_i + t) \prod_{j=1}^{n} \Gamma^{\epsilon_j}(b_j - t),$$

where $m, n \in \mathbb{Z}^+$, $\epsilon_i, \epsilon_j = \pm 1$, and $a_i, b_j, t \in \mathbb{C}$.

The gamma function $\Gamma(t)$ is never equal to 0 and has simple poles when $t = -n$ for $n = 0, 1, 2, \ldots$. Thus, the function $\Gamma^{\epsilon_i}(a_i + t)$ has simple poles only if $\epsilon_i = 1$, in which case the simple poles are at $t = -a_i - n$, for $n = 0, 1, 2, \ldots$. In the complex plane, these poles lie on a horizontal half-line that starts at $-a_i$ and is directed to the left (*i.e.*, in the opposite direction to the positive real axis). Similarly, the function $\Gamma^{\epsilon_j}(b_j - t)$ has simple poles if $\epsilon_j = 1$, in which case the simple poles are at $t = b_j + n$, for $n = 0, 1, 2, \ldots$. In the complex plane, these poles lie on a horizontal half-line which starts at b_j and is directed to the right.

For such a function $f(t)$, a *Barnes integral* is an integral of the form

$$\int_t f(t)\mathrm{d}t,$$

where the path of integration is along the imaginary axis, indented as necessary to ensure that any poles of $\prod_{i=1}^{m} \Gamma^{\epsilon_i}(a_i + t)$ are to the left of the contour and any poles of $\prod_{j=1}^{n} \Gamma^{\epsilon_j}(b_j - t)$ are to the right of the contour. Such a path of integration always exists, provided that for $1 \leq i \leq m$ and $1 \leq j \leq n$ we have $a_i + b_j \neq 0, -1, -2, \ldots$ whenever $\epsilon_i = \epsilon_j = 1$.

In this thesis, whenever we write an integral of the form $\int_t f(t) \mathrm{d}t$, we will always intend it as a Barnes integral with a path of integration as described above.

1.3 Notation and Conventions

Since many of our results involve the products of numerous gamma functions, the following notations (adapted from those introduced by [24]) will prove indispensable, both for compactness and clarity of notation and for better illustrating the symmetries that appear in the relations.

We will use the shorthand

$$\Gamma[a_1, a_2, \ldots, a_m] = \prod_{i=1}^{m} \Gamma(a_i) \tag{1.3.1}$$

and will occasionally extend this to quotients by writing

$$\Gamma \begin{bmatrix} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{bmatrix} = \frac{\Gamma[a_1, \ldots, a_m]}{\Gamma[b_1, \ldots, b_n]} = \frac{\prod_{i=1}^{m} \Gamma(a_i)}{\prod_{j=1}^{n} \Gamma(b_j)}. \tag{1.3.2}$$

To illustrate the symmetries of our relations, we will also write

$$\Gamma[x \pm (a_1, a_2, \ldots, a_m)] = \prod_{i=1}^{m} \Gamma(x \pm a_i) \tag{1.3.3}$$

and

$$\Gamma[x \pm ((a_1, a_2, \ldots, a_m))] = \prod_{\substack{i,j \in \{1,\ldots,m\} \\ i<j}} \prod \Gamma(x \pm a_i \pm a_j) \tag{1.3.4}$$

where the $\pm$ will either always denote $+$ or always denote $-$ in each of its appearances in either (1.3.3) or (1.3.4). Note that the double product in (1.3.4) runs over the set of all $\binom{m}{2}$ two-element subsets of $\{1, \ldots, m\}$.

For example,

$$\Gamma[a + (b, c, d - e)] = \Gamma(a + b)\Gamma(a + c)\Gamma(a + d - e)$$

and

$$\Gamma[a - ((b, c, d, e))] = \Gamma(a - b - c)\Gamma(a - b - d)\Gamma(a - b - e)\Gamma(a - c - d)\Gamma(a - c - e)\Gamma(a - d - e).$$

We will also combine the notation (1.3.1) with the notations (1.3.3) and (1.3.4), so that, for example

$$\Gamma[a, b, c-(e, f, g), d+((e, f, g))] = \Gamma(a)\Gamma(b)\Gamma(c-e)\Gamma(c-f)\Gamma(c-g)\Gamma(d+e+f)\Gamma(d+e+g)\Gamma(d+f+g).$$

Since the gamma function can be related to the sine function by

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},$$

we will also extend the notations (1.3.1)–(1.3.4) to the sine function by replacing each instance of Γ by $\sin \pi$, so that, for example,

$$\sin \pi[a, b+(c, d, e)] = \sin \pi a \sin \pi(b+c) \sin \pi(b+d) \sin \pi(b+e).$$

Finally, note that the sine function has the elementary functional equations

$$\begin{aligned} \sin(-x) &= -\sin(x) \\ \sin(x \pm \pi) &= -\sin(x) \\ \sin(x \pm 2\pi) &= \sin(x) \end{aligned} \tag{1.3.5}$$

of which we will make frequent and sometimes implicit use.

1.4 Objectives

In this thesis, we will consider the function

$$J(a; b; c, d, e, f, g, h) = \frac{I(a; b; c, d, e, f, g, h)}{\Gamma[b, c, d, e, f, g, h, b-a+(c, d, e, f, g, h)]}$$

where

$$I(a; b; c, d, e, f, g, h) = \frac{1}{2\pi\imath}\int_t \frac{\Gamma[a+t, 1+\frac{1}{2}a+t, t+(b, c, d, e, f, g, h), b-a-t, -t]}{\Gamma[\frac{1}{2}a+t, 1+a+t-(c, d, e, f, g, h)]}\mathrm{d}t$$

and $a, b, c, d, e, f, g, h \in \mathbb{C}$ such that $2+3a = b+c+d+e+f+g+h$.

In Chapter 2, we will show that the J function may be expressed as a linear combination of two very well-poised ${}_9F_8(1)$ hypergeometric series, which are complementary with respect to the parameter b. We also present some invariance relations (or, two-term relations) for the J function.

In Chapter 3, we describe these two-term relations for the J function within the context of group theory. In Chapter 4, we derive three-term relations for the J function and also describe these within the context of group theory.

We find 51840 two-term relations satisfied by the J function, which are given by an invariance group G_J that is a subgroup of $GL(8,\mathbb{C})$ and that is isomorphic to the Coxeter group $W(E_6)$ of order 51840. The two-term relations are characterized by a double-coset decomposition of G_J with respect to the subgroup Σ_6 consisting of all of the permutation matrices of G_J.

We then construct a larger subgroup M_J of $GL(8,\mathbb{C})$, which is isomorphic to the Coxeter group $W(E_7)$ of order 2903040 and which contains G_J as a subgroup. We show that for every $\mu_1, \mu_2, \mu_3 \in M_J$, such that μ_1, μ_2, and μ_3 lie in different right cosets of G_J in M_J, there exists a three-term relation involving $J(\mu_1\vec{x})$, $J(\mu_2\vec{x})$, and $J(\mu_3\vec{x})$ where $\vec{x} = (a,b,c,d,e,f,g,h)^T$ and the coefficients of the J functions are rational combinations of sine and gamma functions whose arguments are $\mathbb{Z}$-affine combinations of a, b, c, d, e, f, g, and h. Since there are $\dfrac{2903040}{51840} = 56$ right cosets of G_J in M_J, this gives us $\binom{56}{3} = 27720$ distinct three-term relations satisfied by the J function.

To classify these 27720 relations, we introduce an isometry on a set of 56 vectors corresponding to the cosets of $G_J\backslash M_J$ and use it to develop a notion of "Euclidean type" such that each type corresponds to an orbit of M_J on a set $(G_J\backslash M_J)^{(3)}$ introduced in Section 4.5. We show that there are five Euclidean types (or, equivalently, five orbits) and provide one prototypical relation for each type. Additionally, we show that every other three-term relation can be obtained from one of these five relations through a change of variables of the form $\vec{x} \mapsto \mu\vec{x}$ for some $\mu \in M_J$, applied to all terms and coefficients of the original three-term relation. We further show how to find this element $\mu \in M_J$ and use the notion of Euclidean type to describe the "complexity" of the coefficients of the three-term relations.

The I function defined above (and upon which the J function is based) was also studied by Bailey [1, Chapter 6]. Additionally, Lievens and Van der Jeugt [18] studied a basic hypergeometric analogue of the J

function, which they call $\Phi(a; b; c, d, e, f, g, h)$. Bailey derives a two-term relation for the I function (that we have restated as Proposition 2.2.1), but does not present this as an invariance relation because he does not normalize the I function as we have normalized the J function. Furthermore, he does not place his results within the context of group theory. Lievens and Van der Jeugt find 51840 two-term relations and 27720 three-term relations, place them within the context of group theory, and note that all of the three-term relations can be obtained from five prototypical relations based on the orbits of a certain group action. For the most part, their analysis of the group structure is based on calculation using computational group theory software [10]. We improve upon their analysis by using group theory to prove a number of structure and representation theorems for the groups G_J and M_J. In particular, we introduce a metric and corresponding notion of Euclidean type to determine the orbits of G_J in M_J and to describe the complexity of the coefficients of the three-term relations, while Lievens and Van der Jeugt rely upon a GAP computation to determine the number of orbits and make no attempt to describe the complexity of the coefficients beyond noting that each orbit has a different complexity.

Our strategy of classifying three-term relations based on the actions of parabolic subgroups of Coxeter groups has previously been employed in [9] and [19] in the context of ${}_4F_3(1)$ hypergeometric series. As very well-poised ${}_9F_8(1)$ hypergeometric series can be transformed into ${}_4F_3(1)$ hypergeometric series via a limiting process, it is possible that our results here extend those of [9] and [19].

Chapter 2

The J Function

In this chapter, we define the J function, our basic object of study. We show that it can be expressed as a linear combination of hypergeometric series and derive some of its invariance relations.

2.1 Definition and Representation as a Linear Combination of Hypergeometric Series

Let $a, b, c, d, e, f, g, h \in \mathbb{C}$ be such that

$$2 + 3a = b + c + d + e + f + g + h. \tag{2.1.1}$$

We define the function

$$J(a; b; c, d, e, f, g, h) = \frac{I(a; b; c, d, e, f, g, h)}{\Gamma[b, c, d, e, f, g, h, b - a + (c, d, e, f, g, h)]} \tag{2.1.2}$$

where

$$I(a; b; c, d, e, f, g, h) = \frac{1}{2\pi\imath} \int_t \frac{\Gamma[a + t, 1 + \frac{1}{2}a + t, t + (b, c, d, e, f, g, h), b - a - t, -t]}{\Gamma[\frac{1}{2}a + t, 1 + a + t - (c, d, e, f, g, h)]} \mathrm{d}t.$$

We can represent $I(a; b; c, d, e, f, g, h)$ (and therefore $J(a; b; c, d, e, f, g, h)$) as a linear combination of two ${}_9F_8$ hypergeometric series, which are complementary with respect to b, as follows.

Proposition 2.1.1. *We have*

$$
\begin{aligned}
&I(a;b;c,d,e,f,g,h)\\
&=\frac{\pi\Gamma[a+1,b,c,d,e,f,g,h]}{2\sin\pi(b-a)\Gamma[1+a-(b,c,d,e,f,g,h)]}\\
&\quad\cdot{}_9F_8\left[\begin{matrix}a,1+a/2,b,c,\dots,h\\ a/2,1+a-b,1+a-c,\dots,1+a-h\end{matrix};1\right]\\
&\quad+\frac{\pi\Gamma[2b-a+1,b,b-a+(c,d,e,f,g,h)]}{2\sin\pi(a-b)\Gamma[1+b-(a,c,d,e,f,g,h)]}\\
&\quad\cdot{}_9F_8\left[\begin{matrix}2b-a,1-a/2+b,b,b-a+c,b-a+d,\dots,b-a+h\\ -a/2+b,1+b-a,1+b-c,1+b-d,\dots,1+b-h\end{matrix};1\right]
\end{aligned}
\tag{2.1.3}
$$

In the proof of Proposition 2.1.1, we will make use of the extension of Stirling's Formula to the complex numbers (see [31, Section 4.42] or [35, Section 13.6]).

Proposition 2.1.2 (Stirling's Formula)**.** *Given any $\delta\in(0,\pi)$, if*

$$-\pi+\delta\le\arg(z)\le\pi-\delta,$$

then

$$\Gamma(a+z)=\sqrt{2\pi}z^{a+z-1/2}e^{-z}(1+O(1/|z|))\ \textit{uniformly as } |z|\to\infty. \tag{2.1.4}$$

We also use the following lemma, which is proved in [19].

Lemma 2.1.3. *For every $\epsilon>0$, there is a constant $K=K(\epsilon)$ such that if $\operatorname{dist}(z,\mathbb{Z})\ge\epsilon$, then*

$$|\sin\pi z|\ge Ke^{\pi|Im(z)|}. \tag{2.1.5}$$

Proof of Proposition 2.1.1. The gamma function $\Gamma(t)$ has simple poles when $t=-n$, $n=0,1,2,\dots$, with

$$\operatorname{Res}_{t=-n}\Gamma(t)=\frac{(-1)^n}{n!}.$$

Therefore, the function $\Gamma(-t)$ has simple poles when $t=n$ and the function $\Gamma(b-a-t)$ has simple poles when $t=n+b-a$, for $n=0,1,2,\dots$.

For $N\ge1$, let C_N be the semicircle of radius ρ_N to the right of the imaginary axis and centered at the origin, with ρ_N chosen such that $\rho_N\to\infty$ as $N\to\infty$ and

$$\epsilon:=\min_{N\in\mathbb{Z}^+}\{\operatorname{dist}(C_N,\mathbb{Z}\cup(\mathbb{Z}+b-a))\}>0.$$

The equation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \tag{2.1.6}$$

implies that

$$\Gamma(-t) = \frac{\pi}{\Gamma(1+t)\sin\pi(1+t)} = -\frac{\pi}{\Gamma(1+t)\sin\pi t}$$

and

$$\Gamma(b-a-t) = \frac{\pi}{\Gamma(1-b+a+t)\sin\pi(b-a-t)}.$$

Therefore,

$$\begin{aligned} f(t) &:= \frac{\Gamma[a+t, 1+a/2+t, t+(b,c,d,e,f,g,h), b-a-t, -t]}{\Gamma[a/2+t, 1+a+t-(c,d,e,f,g,h)]} \\ &= \frac{-\pi^2\Gamma[a+t, 1+a/2+t, t+(b,c,d,e,f,g,h)]}{\Gamma[a/2+t, 1+a+t-(c,d,e,f,g,h), 1+t, 1-b+a+t]\sin\pi[t, b-a-t]}. \end{aligned}$$

By Stirling's formula (Proposition 2.1.2),

$$\frac{\Gamma[a+t, 1+a/2+t, t+(b,c,d,e,f,g,h)]}{\Gamma[a/2+t, 1+a+t-(c,d,e,f,g,h), 1+t, 1-b+a+t]} \sim$$

$$t^{a+1+a/2+b+c+d+e+f+g+h-(a/2+1+a-c+1+a-d+1+a-e+1+a-f+1+a-g+1+a-h+1+1-b+a)} = t^{-3}$$

by the hyperplane relation (2.1.1).

By Lemma 2.1.3, there exists a constant $K = K(\epsilon)$ such that

$$\frac{1}{|\sin\pi t \sin\pi(b-a-t)|} \leq \frac{1}{K^2} \text{ if } t \in C_N, N = 1, 2, \ldots.$$

These estimates show that there is a constant $\widetilde{K} > 0$ such that

$$|f(t)| \leq \frac{\widetilde{K}}{|t|^3} \text{ if } t \in C_N, N = 1, 2, \ldots.$$

Thus,

$$\left|\int_{C_N} f(t)\mathrm{d}t\right| \leq \frac{\widetilde{K}}{\rho_N^3} \cdot \pi\rho_N \to 0 \text{ as } N \to \infty,$$

which implies that

$$\int_{C_N} f(t)\mathrm{d}t \to 0 \text{ as } N \to \infty.$$

Thus, the integral given by $I(a;b;c,d,e,f,g,h)$ is equal to the sum of the residues of $f(t)$ at the poles of $\Gamma(-t)$ and $\Gamma(b-a-t)$. Adding up the residues, and making use of the formulas

$$\Gamma(b-a-n)=\frac{\pi}{\sin\pi(b-a-n)\Gamma(1-(b-a-n))}=\frac{(-1)^n\pi}{\sin\pi(b-a)\Gamma(1+a-b+n)}$$

and

$$\Gamma(-b+a-n)=\frac{\pi}{\sin\pi(a-b-n)\Gamma(1-(a-b-n))}=\frac{(-1)^n\pi}{\sin\pi(a-b)\Gamma(1-a+b+n)},$$

we obtain

$$\begin{aligned}
&I(a;b;c,d,e,f,gh)\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma[a+n,1+a/2+n,n+(b,c,d,e,f,g,h),b-a-n]}{n!\Gamma[a/2+n,1+a+n-(c,d,e,f,g,h)]}\\
&\quad+\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma[2b+a-n,b+n,1+a/2+b-a+n,b-a+n+(c,d,e,f,g,h),-b+a-n]}{n!\Gamma[-a/2+b+n,1+b+n-a,1+b+n-(c,d,e,f,g,h)]}\\
&=\sum_{n=0}^{\infty}\frac{\pi\Gamma[a+n,1+a/2+n,n+(b,c,d,e,f,g,h)]}{n!\Gamma[a/2+n,1+a+n-(b,c,d,e,f,g,h)]\sin\pi(b-a)}\\
&\quad+\sum_{n=0}^{\infty}\frac{\pi\Gamma[2b+a-n,b+n,1+a/2+b-a+n,b-a+n+(c,d,e,f,g,h)]}{n!\Gamma[-a/2+b+n,1+b+n-a,1+b+n-(b,c,d,e,f,g,h)]\sin\pi(a-b)}\\
&=\frac{\pi\Gamma[a,1+a/2,b,c,d,e,f,g,h]}{\sin\pi(b-a)\Gamma[a/2,1+a-(b,c,d,e,f,g,h)]}\\
&\quad\cdot\sum_{n=0}^{\infty}\frac{(a)_n(1+a/2)_n(b)_n\ldots(h)_n}{n!(a/2)_n(1+a-b)_n(1+a-c)_n\ldots(1+a-h)_n}\\
&\quad+\frac{\pi\Gamma[2b-a,1-a/2+b,b,b-a+(c,d,e,f,g,h)]}{\sin\pi(a-b)\Gamma[-a/2+b,1+b-a,1+b-(c,d,e,f,g,h)]}\\
&\quad\cdot\sum_{n=0}^{\infty}\frac{(2b-a)_n(1-a/2+b)_n(b)_n(c+b-a)_n(d+b-a)_n\ldots(h+b-a)_n}{n!(-a/2+b)_n(1+b-a)_n(1+b-c)_n(1+b-d)_n\ldots(1+b-h)_n}\\
&=\frac{\pi\Gamma[a,1+a/2,b,c,d,e,f,g,h]}{\sin\pi(b-a)\Gamma[a/2,1+a-(b,c,d,e,f,g,h)]}\\
&\quad\cdot{}_9F_8\left[\begin{matrix}a,1+a/2,b,c,\ldots,h\\a/2,1+a-b,1+a-c,\ldots,1+a-h\end{matrix};1\right]\\
&\quad+\frac{\pi\Gamma[2b-a,1-a/2+b,b,b-a+(c,d,e,f,g,h)]}{\sin\pi(a-b)\Gamma[-a/2+b,1+b-a,1+b-(c,d,e,f,g,h)]}\\
&\quad\cdot{}_9F_8\left[\begin{matrix}2b-a,1-a/2+b,b,b-a+c,b-a+d,\ldots,b-a+h\\-a/2+b,1+b-a,1+b-c,1+b-d,\ldots,1+b-h\end{matrix};1\right].
\end{aligned}$$

We obtain (2.1.3) upon noting that

$$\frac{\Gamma[a,1+a/2]}{\Gamma(a/2)}=\frac{a\Gamma(a)}{2}=\frac{\Gamma(a+1)}{2}$$

and similarly

$$\frac{\Gamma[2b-a, 1-a/2+b]}{\Gamma(-a/2+b)} = \frac{\Gamma(2b-a+1)}{2}.$$

Note that each of the hypergeometric series converges absolutely since the sum of the terms in the denominator minus the sum of the terms in the numerator is $2 > 0$ in each case. □

Remark 2.1.4. Since $J(a;b;c,d,e,f,g,h)$ is a multiple of $I(a;b;c,d,e,f,g,h)$, this shows that we can write $J(a;b;c,d,e,f,g,h)$ as the sum of two ${}_9F_8$ hypergeometric series, the first well-poised in $1+a$ and the second well-poised in $1+2b-a$. The parameters of the second series are obtained from those of the first series by adding $b-a$ to each and then transposing the first two terms, so the two series are complementary with respect to the parameter b.

If we write

$$V(a;b,c,d,e,f,g,h) = \frac{\pi\Gamma[1+a,b,c,d,e,f,g,h]}{2\Gamma[1+a-(b,c,d,e,f,g,h)]} \cdot {}_9F_8\left[\begin{matrix} a, 1+a/2, b, c, \ldots, h \\ a/2, 1+a-b, 1+a-c, \ldots, 1+a-h \end{matrix}; 1\right],$$

then

$$I(a;b;c,d,e,f,g,h) = \frac{V(a;b,c,d,e,f,g,h) - V(2b-a;b,b-a+(c,d,e,f,g,h))}{\sin\pi(b-a)}$$

and

$$J(a;b;c,d,e,f,g,h) = \frac{V(a;b,c,d,e,f,g,h) - V(2b-a;b,b-a+(c,d,e,f,g,h))}{\sin\pi(b-a)\Gamma[b,c,d,e,f,g,h,b-a+(c,d,e,f,g,h)]}.$$

2.2 Invariance Relations

From the definition of the J function, it is clear that J is invariant under permutations of the variables c, d, e, f, g, and h. We derive another invariance relation from Bailey's transformation.

Proposition 2.2.1 (Bailey's transformation)**.** *Let $a,b,c,d,e,f,g,h \in \mathbb{C}$ be such that the hyperplane relation* (2.1.1) *holds. Let $k = 1+2a-c-d-e$. Then*

$$I(a;b;c,d,e,f,g,h) = \Gamma\left[\begin{matrix} c, d, e, f+b-a, g+b-a, h+b-a \\ k+c-a, k+d-a, k+e-a, 1+a-g-h, 1+a-f-h, 1+a-f-g \end{matrix}\right] \cdot I(k,b,k+c-a,k+d-a,k+e-a,f,g,h). \tag{2.2.1}$$

Proof. See [1, Chapter 6]. ☐

If we write

$$\widehat{I}(a;b;c,d,e,f,g,h) = \frac{I(a;b;c,d,e,f,g,h)}{\Gamma[c,d,e,f+b-a,g+b-a,h+b-a]},$$

then (2.2.1) can be rewritten in the form

$$\widehat{I}(a;b;c,d,e,f,g,h) = \widehat{I}(1+2a-c-d-e;b;1+a-d-e,1+a-c-e,1+a-c-d,f,g,h).$$

Since $\Gamma[b,f,g,h,c+b-a,d+b-a,e+b-a]^{-1}$ is invariant under the transformation

$$(a,b,c,d,e,f,g,h) \mapsto (1+2a-c-d-e,b,1+a-d-e,1+a-c-e,1+a-c-d,f,g,h),$$

this also implies that

$$J(a;b;c,d,e,f,g,h) = J(1+2a-c-d-e;b;1+a-d-e,1+a-c-e,1+a-c-d,f,g,h).$$

Chapter 3

Two-Term Relations

In this chapter, we describe the two-term relations for J found in Section 2.2 using group theory. We characterize the transformations involved in these relations according to a notion of "type" developed below and use this notion to determine the isomorphism type of the group of these transformations.

3.1 The Invariance Group G_J

We have shown that $J(a;b;c,d,e,f,g,h)$ is invariant under permutations of c, d, e, f, g, and h and under the transformation

$$(a,b,c,d,e,f,g,h) \mapsto (1+2a-c-d-e, b, 1+a-d-e, 1+a-c-e, 1+a-c-d, f, g, h).$$

Consider the affine hyperplane

$$V = \{(a,b,c,d,e,f,g,h)^T \in \mathbb{C}^8 \colon 2+3a = b+c+d+e+f+g+h\}. \tag{3.1.1}$$

If $\vec{x} = (a,b,c,d,e,f,g,h)^T \in V$, we will write

$$J(\vec{x}) = J(a;b;c,d,e,f,g,h).$$

Let $GL(8,\mathbb{C})$ be the group of invertible 8×8 matrices with complex entries and S_8 be the permutation group acting on eight elements on the left. If $\sigma \in S_8$, we write E_σ for the element of $GL(8,\mathbb{C})$ obtained by permuting the eight rows of the identity matrix $I_8 \in GL(8,\mathbb{C})$ by σ. So, for example,

$$E_{(123)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Matrices of the form E_σ are called *permutation matrices*.

The invariance of J under permutations of c, d, e, f, g, and h implies that

$$J(E_\sigma \vec{x}) = J(\vec{x}) \text{ for any } \sigma \in \langle (34), (45), (56), (67), (78) \rangle.$$

If we define

$$A = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in GL(8,\mathbb{C}), \tag{3.1.2}$$

the invariance of J under the transformation

$$(a,b,c,d,e,f,g,h) \mapsto (1+2a-c-d-e, b, 1+a-d-e, 1+a-c-e, 1+a-c-d, f, g, h)$$

implies (via the hyperplane relation $2+3a = b+c+d+e+f+g+h$) that

$$J(A\vec{x}) = J(\vec{x}).$$

Thus, if we define

$$G_J = \langle E_{(34)}, E_{(45)}, E_{(56)}, E_{(67)}, E_{(78)}, A\rangle \leq GL(8, \mathbb{C}), \tag{3.1.3}$$

then G_J is an invariance group for the function $J(a; b; c, d, e, f, g, h)$ in the sense that for all $M \in G_J$ we have $J(M\vec{x}) = J(\vec{x})$.

3.2 Types of Invariance Relations and the Isomorphism Type of G_J

Note that not all of the invariance relations arising from G_J are essentially "different," in the sense that some relations can be obtained from other relations by some permutation of the last six variables (which corresponds to permuting the rows of the associated matrix in G_J) or by some permutation of the variables c, d, e, f, g, and h (which corresponds to permuting the columns of the associated matrix in G_J). It is useful to construct a minimal set of relations from which all others may be obtained in this fashion.

Definition 3.2.1. Let G be a subgroup of $GL(8, \mathbb{C})$. Let $M_1, M_2 \in G$. We will say that M_1 and M_2 are of the *same type in* G if $M_2 = E_\sigma M_1 E_\tau$, for some permutation matrices $E_\sigma, E_\tau \in G$. If M_1 and M_2 are not of the same type in G, we will say that M_1 and M_2 are of *different type in* G.

Note that for $M \in GL(8, \mathbb{C})$ and $E_\sigma \in GL(8, \mathbb{C})$ a permutation matrix, the product $E_\sigma M$ permutes the rows of M according to σ and ME_σ permutes the columns of M according to σ. Thus, according to the above definition, matrices $M_1, M_2 \in G$ are of the same type in G if and only if we can obtain M_2 by permuting the rows and columns of M_1 using permutation matrices from G.

Definition 3.2.2. For $M \in G$, we define the *type* of M in G to be the set

$$\mathcal{O}_M = \{M' \colon M' \text{ and } M \text{ are of the same type in } G\}.$$

Clearly, the notion of same type is an equivalence relation on G, so the collection of all distinct types $\mathcal{O}_M$ for $M \in G$ forms a partition of G corresponding to a double coset decomposition of G with respect to the subgroup Σ consisting of all the permutation matrices of G. Thus, the type of M in G is the double coset $\Sigma M \Sigma$.

By listing one matrix from each of the distinct types of matrices in G_J, we can find a representative of each of the "different" invariant transformations of J in the above sense.

Proposition 3.2.3. *Let*

$$\Sigma_6 = \{E_\sigma \colon \sigma \in \langle (34), (45), (56), (67), (78) \rangle\} \leq G_J.$$

Then Σ_6 is isomorphic to S_6 and consists of all the permutation matrices in G_J. Furthermore, there are five distinct types of matrices in G_J and Tables 3.1–3.5 provide a representative matrix of each type, along with the number of matrices belonging to that type and the invariance relation that arises from the representative matrix.

Table 3.1: The First Type of Matrix in G_J

$$A_1 = I_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.1}$$

Number of elements $= 1 \cdot 720$
Invariance relation:

$$J(a; b; c, d, e, f, g, h) = J(a; b; c, d, e, f, g, h)$$

Table 3.2: The Second Type of Matrix in G_J

$$A_2 = A = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.2}$$

Number of elements $= 20 \cdot 720$
Invariance relation:

$$J(a; b; c, d, e, f, g, h) = \\ J(1 + 2a - c - d - e; b; 1 + a - d - e, 1 + a - c - e, 1 + a - c - d, f, g, h)$$

Table 3.3: The Third Type of Matrix in G_J

$$A_3 = (E_{(46)(57)}A)^2 E_{(45)(67)} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.3}$$

Number of elements = $30 \cdot 720$
Invariance relation:

$$J(a;b;c,d,e,f,g,h) = J(b-c+h;b;b-a+h,1+a-c-d,1+a-c-e,1+a-c-f, 1+a-c-g,h)$$

Table 3.4: The Fourth Type of Matrix in G_J

$$A_4 = (E_{(36)(47)(58)}A)^2 = \begin{pmatrix} -1/2 & 3/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.4}$$

Number of elements = $20 \cdot 720$
Invariance relation:

$$J(a;b;c,d,e,f,g,h) = J(1+a+b-c-d-e;b;1+a-d-e,1+a-c-e,1+a-c-d, b-a+f,b-a+g,b-a+h)$$

Table 3.5: The Fifth Type of Matrix in G_J

$$A_5 = (E_{(38)}A_3)^2 E_{(38)} = \begin{pmatrix} -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.5}$$

Number of elements $= 1 \cdot 720$
Invariance relation:

$$J(a; b; c, d, e, f, g, h) = J(2b - a; b; b - a + c, b - a + d, b - a + e, b - a + f, b - a + g, b - a + h)$$

In the process of proving this proposition, we will also determine the isomorphism type of G_J.

Definition 3.2.4. The *Dynkin diagram* $G(E_6)$ *of type* E_6 is a graph with vertex set

$$V(E_6) = \{2, 3, 4, 5, 6, 3'\}.$$

Two vertices in the subset $\{2, 3, 4, 5, 6\}$ are adjacent if and only if $|i - j| = 1$, and the vertex $3'$ is adjacent only to 4.

If i and j are any two vertices in $V(E_6)$, we define the integer

$$m(i, j) = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } i \text{ is not adjacent to } j, \\ 3 & \text{if } i \text{ is adjacent to } j. \end{cases} \tag{3.2.6}$$

The *Coxeter group* $W(E_6)$ *of type* E_6 is the group given by the presentation

$$\langle s_2, s_3, s_4, s_5, s_6, s_{3'} : (s_i s_j)^{m(i,j)} = 1 \rangle.$$

Proposition 3.2.5. *The group* G_J *is isomorphic to* $W(E_6)$*. Furthermore, the generators* s_2*,* s_3*,* s_4*,* s_5*,* s_6*, and* $s_{3'}$ *of* $W(E_6)$ *may be identified respectively with the matrices* $E_{(34)}$*,* $E_{(45)}$*,* $E_{(56)}$*,* $E_{(67)}$*,* $E_{(78)}$*, and* A*.*

Proof of Propositions 3.2.3 and 3.2.5. We directly compute the matrices

$$A_3 = (E_{(46)(57)}A)^2 E_{(45)(67)}, A_4 = (E_{(36)(47)(58)}A)^2, A_5 = (E_{(38)}A_3)^2 E_{(38)} \in G_J$$

and see that they are as given in Proposition 3.2.3. We also see that the given invariance relations arise from multiplying the representative matrix by the vector $(a, b, c, d, e, f, g, h)^T \in \mathbb{C}^8$, where $2 + 3a = b + c + d + e + f + g + h$.

Notice that the subgroup $\Sigma_6 = \{E_\sigma : \sigma \in \langle (34), (45), (56), (67), (78) \rangle\}$ of G_J is isomorphic to S_6 and so contains $6! = 720$ elements. Additionally, the action of Σ_6 on G_J by matrix multiplication leaves the upper-left entry of the matrices of G_J unaltered. Since the upper-left entries of the matrices A_1, A_2, A_3, A_4, and A_5 are all distinct, we see that they belong to different types in G_J. We will first obtain lower bounds on the number of matrices of each of the types A_1, A_2, A_3, A_4, and A_5.

Observe that the matrix $A_1 = I_8$ is the identity element of G_J, so that $\Sigma_6 A_1 \Sigma_6 = \Sigma_6$ and so there are at least $|\Sigma_6| = 720$ matrices of the type of A_1 in G_J.

Next, consider the matrix

$$A_2 = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_J.$$

We see that all of the rows of A_2 are distinct as sequences. Therefore, multiplying A_2 on the left by E_σ, for $E_\sigma \in \Sigma_6$, will give us 720 matrices in G_J that belong to the type of A_2. Note that the products $E_\sigma A_2$, for $E_\sigma \in \Sigma_6$, amount to obtaining all possible permutations of the last six rows of A_2. By considering products of the form AE_σ, for $E_\sigma \in \Sigma_6$, we can permute the last six columns of A_2 in every possible way. If we first permute the columns of A_2 that are different as multisets and then permute the rows of the resulting matrix in all 720 different ways, we will obtain 720 new elements of G_J that belong to the type of A_2. Now, the third, fourth, and fifth columns of A_2 are equal as multisets, and the sixth, seventh, and eighth columns of A_2 are equal as a different multiset. Thus, we may permute the last six columns in $\frac{6!}{3!3!} = 20$ different ways.

So, if we permute the columns of A_2 in 20 different ways and then permute the rows of each of the resulting matrices in 720 different ways, we see that the number of matrices belonging to the type of A_2 in G_J is at least $20 \cdot 720$.

Next, consider

$$A_3 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_J.$$

We see that all of the rows of A_3 are distinct as sequences and that the third and eighth columns of A_3 are distinct as multisets while the fourth, fifth, sixth, and seventh columns of A_3 are equal as multisets. Thus, we may permute the columns of A_3 in $\frac{6!}{4!} = 30$ different ways and then permute the rows of each of the resulting matrices in 720 ways, giving at least $30 \cdot 720$ matrices belonging to the type of A_3 in G_J.

Next, consider

$$A_4 = \begin{pmatrix} -1/2 & 3/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_J.$$

Again, the rows of A_4 are distinct as sequences. We see that the third, fourth, and fifth columns are equal as multisets and the sixth, seventh, and eighth columns are equal as multisets, so we may permute the columns of A_4 in $\frac{6!}{3!3!} = 20$ different ways and then permute the rows of each of the resulting matrices in 720 ways, giving at least $20 \cdot 720$ matrices belonging to the type of A_4 in G_J.

Finally, consider

$$A_5 = \begin{pmatrix} -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_J.$$

As above, the rows of A_5 are distinct as sequences. We see that the third through eighth columns are all equal as multisets, so that we permute the columns of A_5 in only one way (*i.e.*, the identity permutation) and then the rows of the matrix in 720 different ways, giving at least $1 \cdot 720$ matrices belonging to the type of A_5 in G_J.

Consider the elements of G_J given by $a_2 = E_{(34)}, a_3 = E_{(45)}, a_4 = E_{(56)}, a_5 = E_{(67)}, a_6 = E_{(78)}$, and $a_{3'} = A$. Clearly, $G_J = \langle a_i \colon i \in \{2, 3, 4, 5, 6, 3'\}\rangle$. We verify by direct calculation that these matrices satisfy $(a_i a_j)^{m(i,j)} = 1$ where $m(i, j)$ is the integer given in Definition 3.2.4.

Therefore, if we define $\varphi(s_i) = a_i$ for each $i \in \{2, 3, 4, 5, 6, 3'\}$, then φ extends uniquely to a surjective homomorphism from $W(E_6)$ onto G_J [8, Section 1.6]. It is well-known (for example, see [16, Section 2.11]) that $W(E_6)$ is a group of order 51840, so to show that φ is in fact an isomorphism, it will be sufficient to show that $|G_J| \geq 51840 = 72 \cdot 720$.

But, we have seen above that G_J contains at least $(1 + 20 + 30 + 20 + 1) \cdot 720 = 51840$ matrices of the types of A_1, A_2, A_3, A_4 and A_5 in G_J, so it must be that $|G_J| = 51840$, $G_J \cong W(E_6)$, and that each of the lower bounds for the number of matrices of the types of A_1, A_2, A_3, A_4 and A_5 in G_J given above are in fact the actual number of matrices of each type in G_J. It then follows that Σ_6 consists of all of the permutation matrices in G_J and thus that we have the complete double coset decomposition $\Sigma_6 M \Sigma_6$, $M \in G_J$, of G_J:

$$G_J = \bigcup_{i=1}^{5} \Sigma_6 A_i \Sigma_6. \qquad \square$$

Remark 3.2.6. As was seen above, the upper-left entry of a matrix is an invariant property of its type in G_J and can be used to identify the types of matrices in G_J.

Chapter 4

Three-Term Relations

Our ultimate goal in this chapter is to find three-term relations of the form

$$\gamma_1(\vec{x})J(\mu_1\vec{x}) + \gamma_2(\vec{x})J(\mu_2\vec{x}) + \gamma_3(\vec{x})J(\mu_3\vec{x}) = 0$$

for matrices μ_1, μ_2, and μ_3 from distinct cosets of the right coset space $G_J\backslash M_J$ of a certain group M_J containing G_J that will be defined in Section 4.2, where the functions $\gamma_1(\vec{x})$, $\gamma_2(\vec{x})$, and $\gamma_3(\vec{x})$ are certain linear combinations of quotients of sine and gamma functions. We achieve this, finding 27720 "essentially different" three-term relations for the J function, in Theorem 4.6.6. To this end, we first find one such three-term relation and define the group M_J as the one generated by the transformations involved in this relation along with the generators of the group G_J. Then, by studying the action of M_J on triples $\{\mu_1, \mu_2, \mu_3\}$ of distinct cosets of $G_J\backslash M_J$ (and, in particular, the orbits of this action), we develop a method for transforming one relation whose transformations $\{\mu_1, \mu_2, \mu_3\}$ come from distinct cosets of $G_J\backslash M_J$ into a new relation involving any other element of the same orbit. Finally, we find one example of a relation coming from each orbit.

4.1 A Three-Term Relation

We will first find one three-term relation for the J-function.

Lemma 4.1.1. *For $a, b, c, t \in \mathbb{C}$, we have*

$$\sin\pi(a-c)\sin\pi(b-a-t) + \sin\pi(b-a)\sin\pi(c-a-t) + \sin\pi(c-b)\sin\pi(-t) = 0. \tag{4.1.1}$$

Proof. Let C^∞ be the vector-space of complex-valued smooth functions and define

$$W = \{f(t) \in C^\infty \colon f'' + \pi^2 f = 0\}.$$

Note that W is a vector space of dimension 2 and that, for any fixed $a \in \mathbb{C}$ and for any $b \in \mathbb{C}$ such that $b-a$ is not an integer, the set

$$\mathcal{B} = \{\sin\pi(b-a-t), \sin\pi(-t)\}$$

is a basis for W [32, Section 19].

We verify directly that $\sin\pi(c-a-t) \in W$, so that

$$\sin\pi(c-a-t) = k_1 \sin\pi(b-a-t) + k_2 \sin\pi(-t) \tag{4.1.2}$$

for some uniquely determined constants $k_1, k_2 \in \mathbb{C}$. Solving the associated system of linear equations, we find that

$$k_1 = -\frac{\sin\pi(a-c)}{\sin\pi(b-a)} \text{ and } k_2 = -\frac{\sin\pi(c-b)}{\sin\pi(b-a)}.$$

(Note that these quantities are well-defined for $b-a \notin \mathbb{Z}$.) Algebraic manipulation of (4.1.2) gives (4.1.1) in the case that $b-a \notin \mathbb{Z}$.

Finally, note that for fixed $a \in \mathbb{C}$, the set of $b \in \mathbb{C}$ such that $b-a \in \mathbb{Z}$ is a set of Lebesgue measure 0, so that, by the continuity of the sine function, (4.1.1) in fact holds for all $b \in \mathbb{C}$. □

Theorem 4.1.2. *For $a, b, c, d, e, f, g, h \in \mathbb{C}$ satisfying* (2.1.1)*, we have the three-term relation*

$$\begin{aligned}
&\sin\pi(b-a)\Gamma\left[b, c, d, e, f, g, h, b-a+(c, d, e, f, g, h)\right] J(a; b; c, d, e, f, g, h) \\
&+\sin\pi(a-c)\Gamma\left[b, c, d, e, f, g, h, c-a+(b, d, e, f, g, h)\right] J(a; c; b, d, e, f, g, h) \\
&+\sin\pi(c-b)\Gamma\left[b, c, b-a+(d, e, f, g, h), c-a+(a, d, e, f, g, h)\right] \\
&\quad\cdot J(2b-a; b+c-a; b, b+d-a, b+e-a, b+f-a, b+g-a, b+h-a) = 0.
\end{aligned}$$

Proof. Let $\vec{x} = (a, b, c, d, e, f, g, h)^T$ and

$$f(\vec{x}, t) = \frac{\Gamma[a+t, 1+\frac{1}{2}a+t, t+(b, c, d, e, f, g, h), b-a-t, -t, c-a-t]}{\Gamma[\frac{1}{2}a+t, 1+a+t-(d, e, f, g, h)]}.$$

By Lemma 4.1.1,

$$\frac{1}{2\pi\imath}\int_t \big[f(\vec{x},t)(\sin\pi(a-c)\sin\pi(b-a-t)$$
$$+\sin\pi(b-a)\sin\pi(c-a-t)+\sin\pi(c-b)\sin\pi(-t))\big]\mathrm{d}t=0. \tag{4.1.3}$$

Note that

$$\Gamma(b-a-t)\sin\pi(b-a-t)=\frac{\pi}{\Gamma(1+a-b+t)},$$
$$\Gamma(c-a-t)\sin\pi(c-a-t)=\frac{\pi}{\Gamma(1+a-c+t)},\ \text{and}$$
$$\Gamma(-t)\sin\pi(-t)=\frac{\pi}{\Gamma(1+t)},$$

so

$$\begin{aligned}&\frac{1}{2\pi\imath}\int_t f(\vec{x},t)\sin\pi(a-c)\sin\pi(b-a-t)\mathrm{d}t\\ =&\frac{\pi\sin\pi(a-c)}{2\pi\imath}\int_t\frac{\Gamma[a+t,1+\frac{1}{2}a+t,t+(b,c,d,e,f,g,h),-t,c-a-t]}{\Gamma[\frac{1}{2}a+t,1+a+t-(b,d,e,f,g,h)]}\mathrm{d}t\\ =&\pi\sin\pi(a-c)I(a;c;b,d,e,f,g,h),\end{aligned}$$

$$\begin{aligned}&\frac{1}{2\pi\imath}\int_t f(\vec{x},t)\sin\pi(b-a)\sin\pi(c-a-t)\mathrm{d}t\\ =&\frac{\pi\sin\pi(b-a)}{2\pi\imath}\int_t\frac{\Gamma[a+t,1+\frac{1}{2}a+t,t+(b,c,d,e,f,g,h),-t,b-a-t]}{\Gamma[\frac{1}{2}a+t,1+a+t-(c,d,e,f,g,h)]}\mathrm{d}t\\ =&\pi\sin\pi(b-a)I(a;b;c,d,e,f,g,h),\ \text{and}\end{aligned}$$

$$\begin{aligned}&\frac{1}{2\pi\imath}\int_t f(\vec{x},t)\sin\pi(c-b)\sin\pi(-t)\mathrm{d}t\\ =&\frac{\pi\sin\pi(c-b)}{2\pi\imath}\int_t\frac{\Gamma[a+t,1+\frac{1}{2}a+t,t+(b,c,d,e,f,g,h),b-a-t,c-a-t]}{\Gamma[\frac{1}{2}a+t,1+a+t-(d,e,f,g,h),1+t]}\mathrm{d}t\\ =&\frac{\pi\sin\pi(c-b)}{2\pi\imath}\int_s\frac{\Gamma[b+s,1+b-a/2+s,b-a+s+(b,c,d,e,f,g,h),-s,c-b-s]}{\Gamma[b-a/2+s,1+b+s-(d,e,f,g,h,a)]}\mathrm{d}s\\ =&\pi\sin\pi(c-b)I(2b-a;b+c-a;b,b+d-a,b+e-a,b+f-a,b+g-a,b+h-a),\end{aligned}$$

where we have let $t\rightsquigarrow b-a+s$ in the third integral and reordered the terms in the final step. Splitting (4.1.3) into the three integrals above and recalling the definition of $J(a;b;c,d,e,f,g,h)$ in (2.1.2) gives the desired result. □

4.2 The Group M_J

Define

$$B = \begin{pmatrix} -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in GL(8, \mathbb{C}) \tag{4.2.1}$$

to be the matrix associated with the transformation

$$(a, b, c, d, e, f, g, h) \mapsto (2b - a, b + c - a, b, b + d - a, b + e - a, b + f - a, b + g - a, b + h - a).$$

Then, the three-term relation in Theorem 4.1.2 can be expressed in the form

$$\gamma_1(\vec{x})J(\vec{x}) + \gamma_2(\vec{x})J(E_{(23)}\vec{x}) + \gamma_3(\vec{x})J(B\vec{x}) = 0.$$

Define

$$M_J = \langle E_{(23)}, B, E_{(34)}, E_{(45)}, E_{(56)}, E_{(67)}, E_{(78)}, A\rangle \leq GL(8, \mathbb{C}),$$

where A is the matrix defined by (3.1.2). Note that $G_J \leq M_J$, so that all of the matrices involved in the two-term and three-term relations that we have found are elements of M_J.

We calculate that $B = A_5 \cdot \big(BE_{(23)}\big)$ and $E_{(23)} = B^2 \cdot \big(BE_{(23)}\big)$ where A_5 is the matrix defined by (3.2.5). Thus,

$$M_J = \langle BE_{(23)}, E_{(34)}, E_{(45)}, E_{(56)}, E_{(67)}, E_{(78)}, A\rangle.$$

Proposition 4.2.1. *The matrix*

$$C = (E_{(34)(56)(78)} B E_{(23)(45)(67)} A)^9 =$$
$$\begin{pmatrix} -5/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \end{pmatrix} \in GL(8, \mathbb{C})$$

is in the center of M_J*.*

Proof. We directly compute the matrix $(E_{(34)(56)(78)} B E_{(23)(45)(67)} A)^9$ and see that it is as given in the statement of the proposition. For $i \in \{2, 3, 4, 5, 6, 7\}$, permuting the i^{th} and $(i+1)^{\text{st}}$ rows of C and the i^{th} and $(i+1)^{\text{st}}$ columns of C yields the same matrix. Furthermore, we calculate that $AC = CA$ and $BC = CB$. Thus, C commutes with all of the generators of M_J and thus with all elements of M_J. □

Via the hyperplane relation $2 + 3a = b + c + d + e + f + g + h$, we see that multiplication by C corresponds to the transformation

$$(a, b, c, d, e, f, g, h) \mapsto (1 - a, 1 - b, 1 - c, 1 - d, 1 - e, 1 - f, 1 - g, 1 - h).$$

Remark 4.2.2. For reference, we compute CA_i for $i = 1, \ldots, 5$ in Table 4.1 below.

Definition 4.2.3. The *Dynkin diagram* $G(E_7)$ *of type* E_7 is a graph with vertex set

$$V(E_7) = \{1, 2, 3, 4, 5, 6, 3'\}.$$

Two vertices in the subset $\{1, 2, 3, 4, 5, 6\}$ are adjacent if and only if $|i - j| = 1$, and the vertex $3'$ is adjacent only to 4.

Table 4.1: The Matrices CA_i for $i = 1, \dots, 5$

$$CA_1 = \begin{pmatrix} -5/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \end{pmatrix}$$

$$CA_2 = \begin{pmatrix} -2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -3/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \end{pmatrix}$$

$$CA_3 = \begin{pmatrix} -3/2 & -1/2 & 3/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \\ -3/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \end{pmatrix}$$

$$CA_4 = \begin{pmatrix} -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -3/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \end{pmatrix}$$

$$CA_5 = \begin{pmatrix} -1/2 & -3/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \end{pmatrix}$$

If i and j are any two vertices in $V(E_7)$, we define the integer

$$m(i,j) = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } i \text{ is not adjacent to } j, \\ 3 & \text{if } i \text{ is adjacent to } j. \end{cases} \tag{4.2.2}$$

The *Coxeter group* $W(E_7)$ *of type* E_7 is the group given by the presentation

$$\langle s_1, s_2, s_3, s_4, s_5, s_6, s_{3'} \colon (s_i s_j)^{m(i,j)} = 1 \rangle.$$

Note that the definition of $m(i,j)$ for E_7 given in (4.2.2) is compatible with and extends the definition of $m(i,j)$ for E_6 given in (3.2.6).

Proposition 4.2.4. *The group* M_J *is isomorphic to* $W(E_7)$*. Furthermore, the generators* s_1*,* s_2*,* s_3*,* s_4*,* s_5*,* s_6*, and* $s_{3'}$ *of* $W(E_7)$ *may be identified respectively with the matrices* $BE_{(23)}$*,* $E_{(34)}$*,* $E_{(45)}$*,* $E_{(56)}$*,* $E_{(67)}$*,* $E_{(78)}$*, and* A*.*

We will prove this proposition and the following proposition simultaneously.

Proposition 4.2.5. *Let*

$$\Sigma_7 = \{E_\sigma \colon \sigma \in \langle (23), (34), (45), (56), (67), (78) \rangle\} \leq M_J.$$

Then Σ_7 *is isomorphic to* S_7 *and consists of all the permutation matrices in* M_J*. Furthermore, there are ten distinct types of matrices in* M_J *with representative matrices given by*

$$\{A_i \colon i = 1, \ldots, 5\} \text{ and } \{CA_i \colon i = 1, \ldots, 5\}. \tag{4.2.3}$$

There are $1 \cdot 5040$ *matrices of each of the types* A_1 *and* CA_1*,* $35 \cdot 5040$ *matrices of each of the types* A_2 *and* CA_2*,* $105 \cdot 5040$ *matrices of each of the types* A_3 *and* CA_3*,* $140 \cdot 5040$ *matrices of each of the types* A_4 *and* CA_4*, and* $7 \cdot 5040$ *matrices of each of the types* A_5 *and* CA_5*.*

Proof of Propositions 4.2.4 and 4.2.5. Notice that the subgroup

$$\Sigma_7 = \{E_\sigma \colon \sigma \in \langle (23), (34), (45), (56), (67), (78) \rangle\}$$

of M_J is isomorphic to S_7 and so contains $7! = 5040$ elements and that Σ_7 leaves the upper-left entry of matrices of M_J unaltered. We saw in Proposition 3.2.3 that the upper-left entries of the matrices A_i, $i = 1, \ldots 5$, are 1, $\frac{1}{2}$, 0, $-\frac{1}{2}$, and -1 respectively and we calculated in Table 4.1 that the upper-left entries of the matrices CA_i, $i = 1, \ldots, 5$, are $-\frac{5}{2}$, -2, $-\frac{3}{2}$, -1, and $-\frac{1}{2}$ respectively. From this consideration, it is clear that all of the matrices A_i and CA_i, $i = 1, \ldots, 5$ are of distinct type in M_J with the exception of the pairs A_4 and CA_5 (which both have upper-left entry $-\frac{1}{2}$) and A_5 and CA_4 (which both have upper-left entry -1). However, since the second row of each matrix A_i is the row vector $\vec{e}_2^{\,T}$ while no row of any CA_i is a row vector of the form $\vec{e}_j^{\,T}$ for $j = 1, \ldots 8$, it follows that all matrices M of type A_i in M_J satisfy $\vec{e}_j^{\,T} M = \vec{e}_k^{\,T}$ for some $2 \leq j, k \leq 8$, while no M matrices of type CA_i in M_J satisfy $\vec{e}_j^{\,T} M = \vec{e}_k^{\,T}$ for any $2 \leq j, k \leq 8$, so that A_4 and CA_5 are of different type in M_J and A_5 and CA_4 are of different type in M_J. Thus, the ten matrices given in (4.2.3) are all of different type in M_J.

Using the same technique as in the proof of Proposition 3.2.3, we obtain a lower bound of $1 \cdot 5040$ matrices of types A_1 and CA_1 in M_J, $\frac{7!}{3!4!} \cdot 5040$ matrices of types A_2 and CA_2, $\frac{7!}{2!4!} \cdot 5040$ matrices of types A_3 and CA_3, $\frac{7!}{3!3!} \cdot 5040$ matrices of types A_4 and CA_4, and $\frac{7!}{6!} \cdot 5040$ matrices of types A_5 and CA_5. Thus, there are at least $2 \cdot (1 + 35 + 105 + 140 + 7) \cdot 5040 = 2903040$ matrices of these types in M_J.

Now, consider the elements of M_J given by $a_1 = BE_{(23)}, a_2 = E_{(34)}, a_3 = E_{(45)}, a_4 = E_{(56)}, a_5 = E_{(67)}, a_6 = E_{(78)}$, and $a_{3'} = A$. Then $M_J = \langle a_i \colon i \in \{1, 2, 3, 4, 5, 6, 3'\}\rangle$. We verify by direct calculation that these matrices satisfy $(a_i a_j)^{m(i,j)} = 1$ where $m(i, j)$ is the integer given in Definition 4.2.3.

Therefore, if we define $\varphi(s_i) = a_i$ for each $i \in \{1, 2, 3, 4, 5, 6, 3'\}$, then φ extends uniquely to a surjective homomorphism from $W(E_7)$ onto M_J as in the proof of Proposition 3.2.3. It is well known (for example, see [16, Section 2.11]) that $W(E_7)$ is a group of order 2903040, so it follows that φ is in fact an isomorphism, as we saw above that $|M_J| \geq 2903040$. Thus, it must be that $|M_J| = 2903040$, $M_J \cong W(E_7)$ and that each of the lower bounds for the number of matrices of the types of A_1, A_2, A_3, A_4, A_5, CA_1, CA_2, CA_3, CA_4, and CA_5 in M_J given above are in fact the actual number of matrices of each type in M_J. It then follows that Σ_7 consists of all of the permutation matrices in M_J and thus that we have the complete double coset decomposition $\Sigma_7 M \Sigma_7$, $M \in M_J$, of M_J:

$$M_J = \left(\bigcup_{i=1}^{5} \Sigma_7 A_i \Sigma_7 \right) \cup \left(\bigcup_{i=1}^{5} \Sigma_7 C A_i \Sigma_7 \right). \qquad \square$$

It is known that the group $W(E_7)$ has a center consisting of two elements [16, Sections 3.20 and 6.4], so the center of M_J is given by the set $\{I_8, C\}$.

4.3 The Coset Space $G_J \backslash M_J$

The group G_J given in Section 3.1 is a subgroup of M_J of index $\frac{2903040}{51840} = 56$. G_J is not a normal subgroup of M_J, as, for example, $E_{(34)} \in G_J$ and $E_{(23)} \in M_J$, but $E_{(23)} E_{(34)} E_{(23)}^{-1} = E_{(23)} E_{(34)} E_{(23)} = E_{(24)} \notin G_J$ (as $E_{(24)} \notin \Sigma_6$ and Σ_6 contains all permutation matrices of G_J, according to Proposition 3.2.3). Nonetheless, we may consider the (right) coset space $G_J \backslash M_J$, which consists of 56 cosets.

Consider the following set of 56 matrices:

$$\mathcal{C}_J = \{C^k E_{(2i)} E_{(23)} B E_{(2j)} \in M_J \colon k \in \{0,1\}, 2 \le i \le j \le 8\},$$

where we adopt the convention that $E_{(22)} = I_8$.

Lemma 4.3.1. *For $2 \le i \le j \le 8$, we have*

$$\vec{e}_2^{\,T} E_{(2i)} E_{(23)} B E_{(2j)} = \begin{cases} \vec{e}_j^{\,T} = \vec{e}_j^{\,T} + \vec{e}_1^{\,T} - \vec{e}_1^{\,T} & \text{if } i = 2, \\ \vec{e}_i^{\,T} + \vec{e}_j^{\,T} - \vec{e}_1^{\,T} & \text{if } i \neq 2,\ i \neq j, \\ \vec{e}_2^{\,T} + \vec{e}_i^{\,T} - \vec{e}_1^{\,T} & \text{if } i \neq 2,\ i = j. \end{cases}$$

Proof. This may be checked by direct calculation. $\square$

Theorem 4.3.2. *The 56 matrices in $\mathcal{C}_J$ lie in different cosets of $G_J \backslash M_J$. Thus,*

$$M_J = \bigcup_{\mu \in \mathcal{C}_J} G_J \mu.$$

Proof. Note that all of the generators of G_J as presented in (3.1.3) have the same second row, $\vec{e}_2^{\,T}$, and so all matrices $M \in G_J$ satisfy $\vec{e}_2^{\,T} M = \vec{e}_2^{\,T}$. Thus, in particular, for any matrices $M \in G_J$ and $N \in M_J$, we have $\vec{e}_2^{\,T} M N = \vec{e}_2^{\,T} N$ and so any two matrices of M_J with different second rows must lie in different cosets of G_J

in M_J. By Lemma 4.3.1, we see that the second rows of the 28 matrices $E_{(2i)}E_{(23)}BE_{(2j)}$, $2 \le i \le j \le 8$ are distinct and we explicitly calculate that the second rows of the 28 matrices $CE_{(2i)}E_{(23)}BE_{(2j)}$ are distinct both from these rows and from each other. Thus, the 56 matrices of $\mathcal{C}_J$ lie in different cosets of $G_J \backslash M_J$, which completes the proof, as we have seen $\frac{|M_J|}{|G_J|} = 56$. □

We have seen that each of the 28 matrices $E_{(2i)}E_{(23)}BE_{(2j)}$, $2 \le i \le j \le 8$, has a second row of the form $\vec{e}_k^T + \vec{e}_\ell^T - \vec{e}_1^T$ for some $1 \le k < \ell \le 8$. Conversely, as there are $\binom{8}{2} = 28$ different rows of this form and each of the matrices $E_{(2i)}E_{(23)}BE_{(2j)}$, $2 \le i \le j \le 8$, has a distinct second row, it follows that for each $1 \le k < \ell \le 8$ there exist $2 \le i \le j \le 8$ such that $\vec{e}_2^T E_{(2i)}E_{(23)}BE_{(2j)} = \vec{e}_k^T + \vec{e}_\ell^T - \vec{e}_1^T$.

Furthermore, we have seen that $\vec{e}_2^T M = \vec{e}_2^T$ for all $M \in G_J$, so that the second row is an invariant property of each coset.

Definition 4.3.3. We will use the notation (k, ℓ) to denote the coset whose matrices have second row $\vec{e}_k^T + \vec{e}_\ell^T - \vec{e}_1^T$ and $(k, \ell)^*$ to denote the coset containing the matrix $CE_{(2i)}E_{(23)}BE_{(2j)}$, where $2 \le i \le j \le 8$ are chosen so that

$$\vec{e}_2^T E_{(2i)}E_{(23)}BE_{(2j)} = \vec{e}_k^T + \vec{e}_\ell^T - \vec{e}_1^T.$$

So, for example, $I_8 \in (1, 2)$ and $CE_{(23)}E_{(23)}BE_{(24)} \in (3, 4)^*$ (by Lemma 4.3.1).

Define the subgroup

$$\Sigma_8 = \langle E_{(23)}B, E_{(23)}, E_{(34)}, E_{(45)}, E_{(56)}, E_{(67)}, E_{(78)} \rangle$$

of M_J. Note that $\Sigma_6 \le \Sigma_7 \le \Sigma_8$. Unlike Σ_6 and Σ_7, the group Σ_8 does not consist entirely of permutation matrices, but the following theorem makes the choice of notation clear.

Theorem 4.3.4. *There is a unique isomorphism of groups*

$$\Phi \colon S_8 \to \Sigma_8$$

such that $\Phi((12)) = E_{(23)}B$ *and* $\Phi(\sigma) = E_\sigma$ *for* $\sigma \in \{(23), (34), (45), (56), (67), (78)\}$.

Proof. We will need the following definition.

Definition 4.3.5. The *Dynkin diagram* $G(A_7)$ *of type* A_7 is a graph with vertex set

$$V(A_7) = \{1, 2, 3, 4, 5, 6, 7\}.$$

Two vertices in the set $\{1, 2, 3, 4, 5, 6, 7\}$ are adjacent if and only if $|i - j| = 1$.

If i and j are any two vertices in $V(A_7)$, we define the integer

$$s(i,j) = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } i \text{ is not adjacent to } j, \\ 3 & \text{if } i \text{ is adjacent to } j. \end{cases} \tag{4.3.1}$$

The *Coxeter group* $W(A_7)$ *of type* A_7 is the group given by the presentation

$$\langle b_1, b_2, b_3, b_4, b_5, b_6, b_7 \colon (b_i b_j)^{s(i,j)} = 1 \rangle.$$

It is well-known that $S_8 \cong W(A_7)$ and that an isomorphism is given by identifying b_1, b_2, b_3, b_4, b_5, b_6, b_7 with (12), (23), (34), (45), (56), (67), and (78) respectively [16, Section 1.1].

Consider the elements $c_1 = E_{(23)}B$, $c_2 = E_{(23)}$, $c_3 = E_{(34)}$, $c_4 = E_{(45)}$, $c_5 = E_{(56)}$, $c_6 = E_{(67)}$ and $c_7 = E_{(78)}$ of Σ_8. Note $\Sigma_8 = \langle c_i \colon i = 1, \ldots, 7 \rangle$. We verify by direct calculation that these matrices satisfy $(c_i c_j)^{s(i,j)} = 1$ where $s(i,j)$ is the integer given in Definition 4.3.5.

Therefore, if we define $\Phi(b_i) = c_i$ for $i = 1, \ldots, 7$, then Φ extends uniquely to a homomorphism from S_8 onto Σ_8 as in the proof of Proposition 3.2.3. But the only normal subgroups of S_8 are S_8, the alternating group A_8, and the trivial group. Since $c_1, \ldots, c_7$ are in the image of Φ, the kernel of Φ must be trivial (as A_8 has index 2 in S_8 and the image contains more than 2 elements) and so Φ is an isomorphism from S_8 to Σ_8. □

Remark 4.3.6. For reference, we compute that

$$\Phi((132)) = \Phi((23)(12)) = E_{(23)}E_{(23)}B = B$$

and

$$\Phi((13)) = \Phi((12)(23)(12)) = E_{(23)}B^2 = BE_{(23)}.$$

4.4 The Action of M_J on $G_J\backslash M_J$

We will see that the action of M_J on $G_J\backslash M_J$ is easier to describe through the use of an isomorphic group $\phi(M_J)$ that we will now define. As in [13, Section 4], we define the vectors $\vec{v}_{i,\,j} = 4(\vec{e}_i+\vec{e}_j)-\sum_{k=1}^{8}\vec{e}_k$ for $1 \leq i < j \leq 8$, so that, for instance, $\vec{v}_{1,2} = (3, 3, -1, -1, -1, -1, -1, -1)^T$. Let $\Omega = \{\pm\vec{v}_{i,\,j} \colon 1 \leq i < j \leq 8\}$. We define a map $\phi\colon G_J\backslash M_J \to \Omega$ by $\phi((i, j)) = \vec{v}_{i,j}$ and $\phi((i, j)^*) = -\vec{v}_{i,j}$ for $1 \leq i < j \leq 8$.

We also define $\phi(s_1) = E_{(13)}$, $\phi(s_2) = E_{(34)}$, $\phi(s_3) = E_{(45)}$, $\phi(s_4) = E_{(56)}$, $\phi(s_5) = E_{(67)}$, $\phi(s_6) = E_{(78)}$, and

$$\phi(s_{3'}) = \begin{pmatrix} 3/4 & 1/4 & -1/4 & -1/4 & -1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 3/4 & 1/4 & 1/4 & 1/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 1/4 & 3/4 & -1/4 & -1/4 & 1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & -1/4 & 3/4 & -1/4 & 1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & -1/4 & -1/4 & 3/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & 1/4 & 1/4 & 3/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 1/4 & 1/4 & 1/4 & -1/4 & 3/4 & -1/4 \\ 1/4 & -1/4 & 1/4 & 1/4 & 1/4 & -1/4 & -1/4 & 3/4 \end{pmatrix},$$

where s_i, $i = 1, \ldots, 6, 3'$ are the generators of $W(E_7)$. We verify by direct calculation that these matrices satisfy $(\phi(s_i)\phi(s_j))^{m(i,j)} = 1$ where $m(i, j)$ is the integer given in Definition 4.2.3. Thus, ϕ extends uniquely to a homomorphism from $W(E_7)$ into $GL(8, \mathbb{C})$. As M_J is isomorphic to $W(E_7)$, we may also think of ϕ as a map $M_J \to GL(8, \mathbb{C})$ via the identifications in the statement of Proposition 4.2.4. That is, $\phi(BE_{(23)}) = E_{(13)}$, $\phi(E_\sigma) = E_\sigma$ for $\sigma \in \{(34), (45), (56), (67), (78)\}$, and $\phi(A) = \phi(s_{3'})$. Throughout the remainder of this thesis, we will use ϕ to refer to the maps $G_J\backslash M_J \to \Omega$, $W(E_7) \to GL(8, \mathbb{C})$, and $M_J \to GL(8, \mathbb{C})$, with the map intended in any given situation clear from the context.

Lemma 4.4.1. *(i) The group $\phi(\Sigma_8)$ consists of all the permutation matrices of $\phi(M_J)$ and, in fact, of all the permutation matrices of $GL(8, \mathbb{C})$ and $\phi(\Sigma_8) \cong S_8$. Furthermore, there are four distinct types in $\phi(M_J)$ with representative matrices I_8, $\phi(A)$, $\phi(C)$, and $\phi(C)\phi(A)$ (where C is the matrix given in Proposition 4.2.1). There are $1 \cdot 40320$ matrices of each of the types of I_8 and $\phi(C)$ in $\phi(M_J)$ and*

$35 \cdot 40320$ *matrices of each of the types of* $\phi(A)$ *and* $\phi(C)\phi(A)$.

(ii) *The type of* I_8 *in* $\phi(M_J)$ *consists of all matrices in* $GL(8,\mathbb{C})$ *for which each row and each column is the multiset* $\{1,0,0,0,0,0,0,0\}$. *The type of* $\phi(A)$ *in* $\phi(M_J)$ *consists of all matrices for which each row and column is the multiset* $\{\frac{3}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\}$. *The type of* $\phi(C)$ *in* $\phi(M_J)$ *consists of all matrices for which each row and column is the multiset* $\{-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\}$. *The type of* $\phi(C)\phi(A)$ *in* $\phi(M_J)$ *consists of all matrices for which each row and column is the multiset* $\{\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},0,0,0,0\}$. *The group* $\phi(M_J)$ *consists of all matrices in* $GL(8,\mathbb{C})$ *for which each row and column is (the same) one of these multisets.*

(iii) *The map* $\phi\colon M_J \to GL(8,\mathbb{C})$ *is injective. That is,* $\phi(M_J) \cong M_J$.

(iv) *The map* $\phi\colon G_J \to GL(8,\mathbb{C})$ *is injective. That is,* $\phi(G_J) \cong G_J$.

Proof. We compute that

$$\phi(C) = \begin{pmatrix} -3/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & -3/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -3/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & -3/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & -3/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & -3/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & -3/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & -3/4 \end{pmatrix}$$

and

$$\phi(C)\phi(A) = \begin{pmatrix} -1/2 & 0 & 1/2 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & -1/2 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 \end{pmatrix}.$$

Note that each row and column of the matrix I_8 is the multiset $\{1, 0, 0, 0, 0, 0, 0, 0\}$, each row and column of the matrix $\phi(A)$ is the multiset $\{\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$, each row and column of the matrix $\phi(C)$ is the multiset $\{-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$, and each row and column of the matrix $\phi(C)\phi(A)$ is the multiset $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0\}$. Thus, I_8, $\phi(A)$, $\phi(C)$, and $\phi(C)\phi(A)$ belong to different types in $\phi(M_J)$.

We verify that $\phi(M) = E_{\Phi^{-1}(M)}$ for $M \in \Sigma_8$, where Φ is the isomorphism $S_8 \to \Sigma_8$ defined in the previous section, by checking this relation on the generators of Σ_8 and noting that ϕ and Φ are homomorphisms. Thus, $\phi(\Sigma_8) = \{E_\sigma : \sigma \in S_8\}$, so that $\phi(M_J)$ contains all permutation matrices of $GL(8, \mathbb{C})$. Thus, $|\phi(\Sigma_8)| = 8! = 40320$. Using the method of Proposition 3.2.3, we obtain the lower bound of $1 \cdot 40320$ matrices in each of the types I_8 and $\phi(C)$ in $\phi(M_J)$. Unfortunately, this method does not give an adequate bound for the types of $\phi(A)$ and $\phi(C)\phi(A)$ in $\phi(M_J)$ (since the columns in each matrix are equal as multisets).

For $\phi(A)$, note that we may, by permuting the columns using matrices from $\phi(\Sigma_8)$, obtain elements of the type of $\phi(A)$ in $\phi(M_J)$ for which the $\frac{3}{4}$ and the three $-\frac{1}{4}$ entries occur in any positions in any given row, giving $8 \cdot \binom{7}{3}$ distinct elements. For each of these, we may permute the second through eighth rows in 7! ways, showing that there are at least $\binom{7}{3} \cdot 8! = 35 \cdot 40320$ elements of the type of $\phi(A)$ in $\phi(M_J)$.

For $\phi(C)\phi(A)$, a parallel argument with the entry $-\frac{1}{2}$ in the place of $\frac{3}{4}$ and the entries $\frac{1}{2}$ in the place of $-\frac{1}{4}$ likewise gives $35 \cdot 40320$ elements of the type of $\phi(C)\phi(A)$ in $\phi(M_J)$.

This shows that $|\phi(M_J)| \geq (1 + 35 + 35 + 1) \cdot 40320 = 2903040$. But since (by definition) ϕ maps surjectively onto $\phi(M_J)$, it must be that $|\phi(M_J)| \leq |M_J| = 2903040$, so that $|\phi(M_J)| = 2903040$, each of the lower bounds for the number of elements of each type in M_J is in fact the number of elements of that type, and we have found the complete double coset decomposition. This completes the proof of part (i), and parts (ii) and (iii) follow immediately. Part (iv) follows from part (iii) by considering the restriction of the map ϕ to G_J. □

Lemma 4.4.2. *(i) Let $M \in M_J$ and $\mu \in G_J \backslash M_J$ be a right coset. Then $\phi(\mu \cdot M) = \phi(M)^{-1} \cdot \phi(\mu)$.*

(ii) For all $\vec{v} \in \Omega$, $\phi(C)\vec{v} = -\vec{v}$.

Proof. (i) We verify this directly for the generators of M_J. Then, by Lemma 4.4.1(iii), this holds for all of M_J.

(ii) This is a restatement of part (i) for the special case $M = C$.

□

The action of the generators of $\phi(M_J)$ on Ω is described in the following propositions.

Definition 4.4.3. Let $\sigma \in S_8$. Given a right coset (i, j) or $(i, j)^*$ in $G_J \backslash M_J$, we will define

$$\sigma \cdot (i, j) = (\sigma(i), \sigma(j)) \text{ and } \sigma \cdot (i, j)^* = (\sigma(i), \sigma(j))^*.$$

We also extend this to an action of S_8 on Ω via the identification $\phi\colon G_J \backslash M_J \to \Omega$.

Proposition 4.4.4. *(i) Let $\mu \in G_J \backslash M_J$ be a right coset and let $\sigma \in S_8$. Then*

$$\mu \cdot \Phi(\sigma) = \sigma^{-1} \cdot \mu$$

where $\Phi\colon S_8 \to \Sigma_8$ is the isomorphism from Theorem 4.3.4.

(ii) Let $\vec{v} \in \Omega$ and let $E_\sigma \in \phi(\Sigma_8)$ for some $\sigma \in S_8$. Then

$$E_\sigma \cdot \vec{v} = \sigma \cdot \vec{v}.$$

Proof. (i) We verify directly that

$$\mu \cdot \Phi(\sigma) = \sigma^{-1} \cdot \mu \text{ for } \sigma \in \{(12), (23), (34), (45), (56), (67), (78)\}.$$

Then, since these permutations generate S_8 and Φ is an isomorphism, this holds for all $\sigma \in S_8$.

(ii) Apply the map ϕ to the equation in (i). The result follows by Lemma 4.4.2(i) and recalling that $\phi(M) = E_{\Phi^{-1}(M)}$ for $M \in \Sigma_8$.

□

Proposition 4.4.5. *The matrix $\phi(A)$ acts on Ω as a bifid transformation. More specifically, if we let $C_1 = \{1, 3, 4, 5\}$ and $C_2 = \{2, 6, 7, 8\}$, then:*

(i) if $i, j \in C_m$ for $m = 1, 2$ and k, ℓ are chosen so that $C_m = \{i, j, k, \ell\}$, then $\phi(A) \cdot \pm\vec{v}_{i,j} = \mp\vec{v}_{k,\ell}$ where $\mp$ denotes the opposite sign of $\pm$, and

(ii) if one of i, j lies in C_1 and the other lies in C_2, then $\phi(A) \cdot \pm\vec{v}_{i,j} = \pm\vec{v}_{i,j}$.

Proof. This can be verified by explicitly computing each of the 28 products $\phi(A) \cdot \vec{v}_{i,j}$ and then using Lemma 4.4.2(ii) to compute $\phi(A) \cdot -\vec{v}_{i,j}$. □

Lemma 4.4.6. *The group $\phi(M_J)$ acts faithfully and transitively on the set Ω.*

Proof. We will first use the double coset decomposition found in Lemma 4.4.1 to verify that $\phi(M_J)$ acts faithfully. Suppose that $M \in \phi(M_J)$. We will show that $M \cdot \vec{v} = \vec{v}$ can hold for all $\vec{v} \in \Omega$ only if $M = I_8$. Note that M must have one of the forms E_σ, $\phi(C)E_\sigma$, $E_\sigma\phi(A)E_\tau$, or $\phi(C)E_\sigma\phi(A)E_\tau$ for some $\sigma, \tau \in S_8$, by Lemma 4.4.1 and noting that $\phi(C)$ is central. We use Propositions 4.4.4(ii) and 4.4.5 to perform the following calculations.

If $M = E_\sigma$ where σ is not the identity permutation, then we may pick $i, j \in \{1, \ldots, 8\}$ such that $\sigma(i) \neq i$ and $\sigma(j) \notin \{i, \phi(i)\}$. Let $\vec{v} = \vec{v}_{i,j}$. Then

$$M\vec{v} = \vec{v}_{\sigma(i),\sigma(j)} \neq \vec{v}.$$

If $M = \phi(C)E_\sigma$, then $\phi(C)E_\sigma\vec{v}_{1,2} = -\vec{v}_{\sigma(1),\sigma(2)} \neq \vec{v}_{1,2}$.

If $M = E_\sigma\phi(A)E_\tau$, let $\vec{v} = \vec{v}_{\tau^{-1}(1),\tau^{-1}(3)}$ so that $E_\tau\vec{v} = \vec{v}_{1,3}$. Then,

$$E_\sigma\phi(A)E_\tau\vec{v} = E_\sigma\phi(A)\vec{v}_{1,3} = E_\sigma(-\vec{v}_{4,5}) = -\vec{v}_{\sigma(4),\sigma(5)} \neq \vec{v}_{\tau^{-1}(1),\tau^{-1}(3)}.$$

If $M = \phi(C)E_\sigma\phi(A)E_\tau$, let $\vec{v} = \vec{v}_{\tau^{-1}(1),\tau^{-1}(2)}$ so that $E_\tau\vec{v} = \vec{v}_{1,2}$. Then,

$$\phi(C)E_\sigma\phi(A)E_\tau\vec{v} = \phi(C)E_\sigma\phi(A)\vec{v}_{1,2} = \phi(C)E_\sigma\vec{v}_{1,2} = \phi(C)\vec{v}_{\sigma(1),\sigma(2)} = -\vec{v}_{\sigma(1),\sigma(2)}$$
$$\neq \vec{v}_{\tau^{-1}(1),\tau^{-1}(2)}.$$

Thus, in all cases, if $M \neq I_8$, then there is some $\vec{v} \in \Omega$ such that $M\vec{v} \neq \vec{v}$ and so the action of $\phi(M_J)$ on Ω is faithful.

Finally, we note that the action of $\phi(M_J)$ on Ω is transitive since $\phi(M_J)$ contains all of the permutation matrices of $GL(8,\mathbb{C})$ and since the central element $\phi(C)$ maps $\vec{v}$ to $-\vec{v}$ for all $\vec{v} \in \Omega$. □

Definition 4.4.7. Let $\vec{v} = (v_1, \ldots, v_8)^T, \vec{w} = (w_1, \ldots, w_8)^T \in \Omega$. We define the *Euclidean distance* between $\vec{v}$ and $\vec{w}$ to be the number $d(\vec{v}, \vec{w}) = \sqrt{\sum_{i=1}^{8}(v_i - w_i)^2}$.

Remark 4.4.8. Note that the distance between any two elements of Ω is one of 0, $\sqrt{32}$, $\sqrt{64}$, and $\sqrt{96}$. Specifically, for distinct $i, j, k, \ell \in \{1, \ldots 8\}$, we have $d(\vec{v}_{i,j}, \vec{v}_{i,j}) = 0$, $d(\vec{v}_{i,j}, \vec{v}_{i,k}) = \sqrt{32}$, $d(\vec{v}_{i,j}, \vec{v}_{k,\ell}) = \sqrt{64}$, $d(\vec{v}_{i,j}, -\vec{v}_{i,j}) = \sqrt{96}$, $d(\vec{v}_{i,j}, -\vec{v}_{i,k}) = \sqrt{64}$, and $d(\vec{v}_{i,j}, -\vec{v}_{k,\ell}) = \sqrt{32}$.

Lemma 4.4.9. *(i) The set Ω is a metric space with respect to Euclidean distance.*

(ii) For any $\vec{v}, \vec{w} \in \Omega$, we have $\vec{v} \cdot \vec{w} = 24 - \frac{1}{2}d(\vec{v}, \vec{w})^2$.

(iii) The action of $\phi(M_J)$ on Ω is by isometries with respect to Euclidean distance.

Proof. Part (i) is a routine exercise using the definitions. Part (ii) may be checked case by case using Remark 4.4.8.

To prove part (iii), we note that the group $\phi(M_J)$ consists of orthogonal matrices, so the action of $\phi(M_J)$ on Ω respects scalar product. By part (ii), this action also respects Euclidean distance. That is to say, for $M \in \phi(M_J)$ and $\vec{v}, \vec{w} \in \Omega$, we have

$$d(M\vec{v}, M\vec{w})^2 = 48 - 2(M\vec{v}) \cdot (M\vec{w}) = 48 - 2\vec{v} \cdot \vec{w} = d(\vec{v}, \vec{w})^2.$$

Since $d(\vec{v}_1, \vec{v}_2) \geq 0$ for all $\vec{v}_1, \vec{v}_2 \in \Omega$, we have

$$d(M\vec{v}, M\vec{w}) = d(\vec{v}, \vec{w}).$$

□

4.5 The Orbits of the Action of M_J on $(G_J\backslash M_J)^{(3)}$

Definition 4.5.1. Let $(G_J\backslash M_J)^{(3)}$ be the subset of the power set of $G_J\backslash M_J$ consisting of all unordered triples $\{a, b, c\}$ of distinct elements of $G_J\backslash M_J$ (*i.e.*, $a \neq b \neq c \neq a$). Let $\Omega^{(3)}$ be the subset of the power set of Ω consisting of all unordered triples $\{a, b, c\}$ of distinct elements of Ω (*i.e.*, $a \neq b \neq c \neq a$).

Although the tuples of $(G_J\backslash M_J)^{(3)}$ and $\Omega^{(3)}$ are unordered, we will always keep the same ordering when writing the elements of a tuple, so that, in particular, $\{a_1, a_2, a_3\} = \{b_1, b_2, b_3\}$ will always mean that $a_i = b_i$ for $i = 1, 2, 3$. We extend the action of M_J on $G_J\backslash M_J$ to an action on $(G_J\backslash M_J)^{(3)}$, as follows.

Proposition 4.5.2. *The group M_J acts on $(G_J\backslash M_J)^{(3)}$ diagonally via*

$$\{a, b, c\} \cdot M = \{a \cdot M, b \cdot M, c \cdot M\}.$$

Proof. This is a routine exercise using the definitions. □

Analogously to Proposition 4.5.2, we extend the action of $\phi(M_J)$ to $\Omega^{(3)}$ diagonally via $g \cdot \{a, b, c\} = \{g \cdot a, g \cdot b, g \cdot c\}$ and we likewise extend the action of S_8 on $G_J\backslash M_J$ and Ω from Definition 4.4.3 to $(G_J\backslash M_J)^{(3)}$ and $\Omega^{(3)}$ diagonally.

By Lemma 4.4.1, the Euclidean distance defined in Definition 4.4.7 also provides a notion of distance for $G_J\backslash M_J$. Namely, if $a, b \in G_J\backslash M_J$, the distance between a and b is given by $d(a, b) = d(\phi(a), \phi(b))$. For example, $d((1, 2), (3, 4)^*) = d(\vec{v}_{1,2}, -\vec{v}_{3,4}) = \sqrt{32}$. By Remark 4.4.8, the distance between any two cosets is one of 0, $\sqrt{32}$, $\sqrt{64}$, and $\sqrt{96}$. Given a triple $\{a, b, c\}$ in $(G_J\backslash M_J)^{(3)}$ or $\Omega^{(3)}$, we are also interested in the unordered multiset of distances $\{d(a, b), d(a, c), d(b, c)\}$, which we will always write in increasing order. To simplify notation, we will in this context write A for $\sqrt{32}$, B for $\sqrt{64}$, and C for $\sqrt{96}$. So, for example, the distance triple $\{\sqrt{32}, \sqrt{32}, \sqrt{32}\}$ will be denoted AAA, and the distance triple $\{\sqrt{32}, \sqrt{64}, \sqrt{96}\}$ will be denoted ABC. If a triple $\{a, b, c\}$ has distance triple xyz (where x, y, and z all come from $\{A, B, C\}$), we will say that the triple is of *Euclidean type xyz*. We will now classify the elements of $(G_J\backslash M_J)^{(3)}$ according to their Euclidean types.

Note that Proposition 4.4.4(i) tells us that any two elements of $(G_J\backslash M_J)^{(3)}$ that lie in the same orbit of the action of S_8 also have the same Euclidean type. So, for example, $\{(1, 2), (3, 4), (5, 6)^*\}$ and

$\{(1,2),(3,5),(4,6)^*\}$ have the same Euclidean type since

$$(45)\cdot\{(1,2),(3,4),(5,6)^*\}=\{(1,2),(3,5),(4,6)^*\}.$$

By replacing the numbers with variables $i, j, k, \ldots$, we obtain a "prototype" of an element of $(G_J\backslash M_J)^{(3)}$ (or, more precisely, an orbit representative for the action of S_8 on $(G_J\backslash M_J)^{(3)}$). Such a prototype is not unique, since for example the triple $\{(1,2),(1,3),(1,4)\}$ can be described as being of either prototype $\{(i,j),(i,k),(i,\ell)\}$ or prototype $\{(i,j),(j,k),(j,\ell)\}$. Nonetheless, we can obtain a list of prototypes such that all elements of $(G_J\backslash M_J)^{(3)}$ can be described as having exactly one prototype from the list. We provide such a list in Table 4.2, along with the number of elements of each Euclidean type. The triples $\{a,b,c\}$ are ordered such that $d(a,b)\le d(a,c)\le d(b,c)$. For example, Table 4.2 shows that there are 840 triples $\{(i,j),(i,k),(j,\ell)\}$ and 840 triples $\{(i,j)^*,(i,k)^*,(j,\ell)^*\}$ belonging to the Euclidean type AAB of size 7560 (corresponding to the distance multiset $\{\sqrt{32},\sqrt{32},\sqrt{64}\}$) and that $d((i,j),(i,k))=d((i,j),(j,\ell))=\sqrt{32}$ and $d((i,k),(j,\ell))=\sqrt{64}$, and $d((i,j)^*,(i,k)^*)=d((i,j)^*,(j,\ell)^*)=\sqrt{32}$ and $d((i,k)^*,(j,\ell)^*)=\sqrt{64}$.

The number of elements of each prototype given in Table 4.2 are computed using standard combinatorial techniques. For example, in prototype $\{(i,j),(i,k),(i,\ell)\}$, there are 8 choices for i and $\binom{7}{3}$ choices for j, k, and ℓ, since the triple is unordered. We obtain the number of triples in each orbit by adding up the number of triples of each type in that orbit. Since these add to $27720=\binom{56}{3}=|(G_J\backslash M_J)^{(3)}|$ triples, we have described all triples of $(G_J\backslash M_J)^{(3)}$ and so have found prototypes for every element of $(G_J\backslash M_J)^{(3)}$. In particular, note that this implies that every element of $(G_J\backslash M_J)^{(3)}$ has one of the five Euclidean types AAA, AAB, ABB, BBB, or ABC.

Lemma 4.5.3. *If two elements $a,b\in\Omega^{(3)}$ both have Euclidean type AAA, then there exists a transformation $w\in\phi(M_J)$ such that $w\cdot a=b$.*

Proof. From Table 4.2 translated to $\Omega^{(3)}$ via the map ϕ, we see that every element of Euclidean type AAA has one of the following six prototypes: $\{\vec{v}_{i,j},\vec{v}_{i,k},\vec{v}_{i,\ell}\}$, $\{\vec{v}_{i,j},\vec{v}_{i,k},\vec{v}_{j,k}\}$, $\{\vec{v}_{i,j},\vec{v}_{i,k},-\vec{v}_{\ell,m}\}$, $\{-\vec{v}_{i,j},-\vec{v}_{i,k},-\vec{v}_{i,\ell}\}$, $\{-\vec{v}_{i,j},-\vec{v}_{i,k},-\vec{v}_{j,k}\}$, $\{-\vec{v}_{i,j},-\vec{v}_{i,k},\vec{v}_{\ell,m}\}$. As the latter three are obtained by negating the first three, it is sufficiently to consider only the first three prototypes in light of Lemma 4.4.2(ii). Also, in light of Proposi-

Table 4.2: Characterizations of the Euclidean types of $(G_J \backslash M_J)^{(3)}$

Type AAA		4032
$\{(i,j),(i,k),(i,\ell)\}$	$\{(i,j)^*,(i,k)^*,(i,\ell)^*\}$	$8 \cdot \binom{7}{3} = 280$
$\{(i,j),(i,k),(j,k)\}$	$\{(i,j)^*,(i,k)^*,(j,k)^*\}$	$\binom{8}{3} = 56$
$\{(i,j),(i,k),(\ell,m)^*\}$	$\{(i,j)^*,(i,k)^*,(\ell,m)\}$	$8 \cdot \binom{7}{2}\binom{5}{2} = 1680$
Type AAB		7560
$\{(i,j),(i,k),(j,\ell)\}$	$\{(i,j)^*,(i,k)^*,(j,\ell)^*\}$	$\binom{8}{2} \cdot 6 \cdot 5 = 840$
$\{(i,j),(i,k),(k,\ell)^*\}$	$\{(i,j)^*,(i,k)^*,(k,\ell)\}$	$8 \cdot 7 \cdot 6 \cdot 5 = 1680$
$\{(i,j)^*,(k,\ell),(m,n)\}$	$\{(i,j),(k,\ell)^*,(m,n)^*\}$	$\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{2} = 1260$
Type ABB		12096
$\{(i,j),(i,k),(\ell,m)\}$	$\{(i,j)^*,(i,k)^*,(\ell,m)^*\}$	$8 \cdot \binom{7}{2}\binom{5}{2} = 1680$
$\{(i,j),(i,k),(j,k)^*\}$	$\{(i,j)^*,(i,k)^*,(j,k)\}$	$8 \cdot \binom{7}{2} = 168$
$\{(i,j),(i,k),(i,\ell)^*\}$	$\{(i,j)^*,(i,k)^*,(i,\ell)\}$	$8 \cdot \binom{7}{2} \cdot 5 = 840$
$\{(i,j),(k,\ell)^*,(k,m)\}$	$\{(i,j)^*,(k,\ell),(k,m)^*\}$	$8 \cdot 7 \cdot \binom{6}{2} \cdot 4 = 3360$
Type BBB		2520
$\{(i,j),(k,\ell),(m,n)\}$	$\{(i,j)^*,(k,\ell)^*,(m,n)^*\}$	$\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{3!} = 420$
$\{(i,j),(k,\ell),(i,k)^*\}$	$\{(i,j)^*,(k,\ell)^*,(i,k)\}$	$\binom{8}{2} \cdot 6 \cdot 5 = 840$
Type ABC		1512
$\{(i,j),(i,k),(i,k)^*\}$	$\{(i,j)^*,(i,k)^*,(i,k)\}$	$8 \cdot 7 \cdot 6 = 336$
$\{(i,j),(k,\ell),(k,\ell)^*\}$	$\{(i,j)^*,(k,\ell)^*,(k,\ell)\}$	$\binom{8}{2}\binom{6}{2} = 420$

tion 4.4.4(ii), it is sufficient to check this for a specific representative of each prototype. These simplification steps will also be used in the four propositions that follow. The computations that follow come from Proposition 4.4.5.

Note that

$$\phi(C)\phi(A) \cdot \{\vec{v}_{3,4}, \vec{v}_{3,5}, \vec{v}_{3,6}\} = \{\vec{v}_{1,4}, \vec{v}_{1,5}, -\vec{v}_{3,6}\}$$

transforms the first prototype into the third and

$$\phi(A) \cdot \{\vec{v}_{1,2}, \vec{v}_{1,6}, -\vec{v}_{7,8}\} = \{\vec{v}_{1,2}, \vec{v}_{1,6}, \vec{v}_{2,6}\}$$

transforms the third prototype into the second. □

Lemma 4.5.4. *If two elements $a, b \in \Omega^{(3)}$ both have Euclidean type AAB, then there exists a transformation $w \in \phi(M_J)$ such that $w \cdot a = b$.*

Proof. After simplifying as in Lemma 4.5.3, it remains to show that we can transform elements between the prototypes $\{\vec{v}_{i,j}, \vec{v}_{i,k}, \vec{v}_{j,\ell}\}$, $\{\vec{v}_{i,j}, \vec{v}_{i,k}, -\vec{v}_{k,\ell}\}$, and $\{-\vec{v}_{i,j}, \vec{v}_{k,\ell}, \vec{v}_{m,n}\}$.

Note that

$$\phi(A) \cdot \{\vec{v}_{1,2}, \vec{v}_{1,6}, \vec{v}_{2,7}\} = \{\vec{v}_{1,2}, \vec{v}_{1,6}, -\vec{v}_{6,8}\}$$

transforms the first prototype into the second and

$$\phi(C)\phi(A) \cdot \{\vec{v}_{1,2}, \vec{v}_{1,3}, \vec{v}_{2,6}\} = \{-\vec{v}_{1,2}, \vec{v}_{4,5}, \vec{v}_{7,8}\}$$

transforms the first prototype into the third. □

Lemma 4.5.5. *If two elements $a, b \in \Omega^{(3)}$ both have Euclidean type ABB, then there exists a transformation $w \in \phi(M_J)$ such that $w \cdot a = b$.*

Proof. After simplifying as in Lemma 4.5.3, it remains to show that we can transform elements between the prototypes $\{\vec{v}_{i,j}, \vec{v}_{i,k}, \vec{v}_{\ell,m}\}$, $\{\vec{v}_{i,j}, \vec{v}_{i,k}, -\vec{v}_{j,k}\}$, $\{\vec{v}_{i,j}, \vec{v}_{i,k}, -\vec{v}_{i,\ell}\}$ and $\{\vec{v}_{i,j}, -\vec{v}_{k,\ell}, \vec{v}_{k,m}\}$.

Note that

$$\phi(A) \cdot \{\vec{v}_{1,2}, \vec{v}_{1,6}, \vec{v}_{7,8}\} = \{\vec{v}_{1,2}, \vec{v}_{1,6}, -\vec{v}_{2,6}\}$$

transforms the first prototype into the second,

$$\phi(A) \cdot \{\vec{v}_{1,2}, \vec{v}_{1,6}, \vec{v}_{4,5}\} = \{\vec{v}_{1,2}, \vec{v}_{1,6}, -\vec{v}_{1,3}\}$$

transforms the first prototype into the third, and

$$\phi(A) \cdot \{\vec{v}_{1,2}, \vec{v}_{1,3}, \vec{v}_{4,6}\} = \{\vec{v}_{1,2}, -\vec{v}_{4,5}, \vec{v}_{4,6}\}$$

transforms the first prototype into the fourth. □

Lemma 4.5.6. *If two elements $a, b \in \Omega^{(3)}$ both have Euclidean type BBB, then there exists a transformation $w \in \phi(M_J)$ such that $w \cdot a = b$.*

Proof. After simplifying as in Lemma 4.5.3, it remains to show that we can transform elements from the prototype $\{\vec{v}_{i,j}, \vec{v}_{k,\ell}, \vec{v}_{m,n}\}$ to the prototype $\{\vec{v}_{i,j}, \vec{v}_{k,\ell}, -\vec{v}_{i,k}\}$.

Note that

$$\phi(A) \cdot \{\vec{v}_{1,2}, \vec{v}_{3,6}, \vec{v}_{4,5}\} = \{\vec{v}_{1,2}, \vec{v}_{3,6}, -\vec{v}_{1,3}\}$$

accomplishes this. □

Lemma 4.5.7. *If two elements $a, b \in \Omega^{(3)}$ both have Euclidean type ABC, then there exists a transformation $w \in \phi(M_J)$ such that $w \cdot a = b$.*

Proof. After simplifying as in Lemma 4.5.3, it remains to show that we can transform elements from the prototype $\{\vec{v}_{i,j}, \vec{v}_{i,k}, -\vec{v}_{i,k}\}$ to the prototype $\{\vec{v}_{i,j}, \vec{v}_{k,\ell}, -\vec{v}_{k,\ell}\}$.

Note that

$$\phi(A) \cdot \{\vec{v}_{1,2}, \vec{v}_{1,3}, -\vec{v}_{1,3}\} = \{\vec{v}_{1,2}, \vec{v}_{4,5}, -\vec{v}_{4,5}\}$$

accomplishes this. □

Theorem 4.5.8. *(i) If $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are two elements of $\Omega^{(3)}$ for which $d(a_i, a_j) = d(b_i, b_j)$ for all $i, j \in \{1, 2, 3\}$, then there exists $w \in \phi(M_J)$ such that $w(a_i) = b_i$ for all $i \in \{1, 2, 3\}$.*

(ii) Two elements of $\Omega^{(3)}$ are in the same $\phi(M_J)$-orbit if and only if they have the same Euclidean type.

(iii) The group $\phi(M_J)$ has precisely 5 orbits in its action on $\Omega^{(3)}$. These correspond to the Euclidean types AAA, AAB, ABB, BBB, and ABC.

Proof. (i) This follows from Lemmas 4.5.3–4.5.7 since every element of $\Omega^{(3)}$ has one of the five prototypes AAA, AAB, ABB, BBB, ABC.

(ii) The "if" direction follows from (i). The "only if" direction follows from the fact, proven in Lemma 4.4.9(iii), that $\phi(M_J)$ acts on Ω by isometries.

(iii) The result follows from (ii) since we have shown that these are the only five Euclidean types and that there are elements of $\Omega^{(3)}$ with each of these five types. □

4.6 Types of Three-Term Relations

The notions of Euclidean distance on $G_J\backslash M_J$ and the classification of unordered triples (μ_1, μ_2, μ_3) of distinct elements of $G_J\backslash M_J$ into Euclidean types, defined in the previous section, will be used below in our study of the three-term relations of $J(a; b; c, d, e, f, g, h)$.

For $\mu \in G_J\backslash M_J$ and V the affine hyperplane of 3.1.1, we define the functions $J_\mu(\vec{x})\colon V \to \mathbb{C}$ by $J_\mu(\vec{x}) = J(\mu\vec{x})$. We call a relation among J_{μ_1}, J_{μ_2}, and J_{μ_3} an "xyz relation" if the triple (μ_1, μ_2, μ_3) is of Euclidean type xyz.

We are now ready to state and prove our main results concerning three-term relations for $J(\vec{x})$. In deriving the three-term relations, we will make frequent use of the following trigonometric identities to simplify the coefficients:

$$\sin\pi[2a, 2b, 2c, 2d] =$$
$$\sin\pi[a + b + c + d - (2a, 2b, 2c, 2d)] - \sin\pi[b + c + d - a - (-2a, 2b, 2c, 2d)] \tag{4.6.1}$$

and

$$\sin\pi[3a,3b,3c,3d,3e,3f,3f-3e] =$$

$$\sin\pi[3e, a+b+c+d-e+2f-(3a,3b,3c,3d,3f,3f-3e)]$$

$$-\sin\pi[3f, a+b+c+d-f+2e-(3a,3b,3c,3d,3e,3e-3f)]$$

$$+\sin\pi[3f-3e, a+b+c+d-e-f-(3a,3b,3c,3d,-3e,-3f)]. \tag{4.6.2}$$

These identities appear in [11, Exercises 2.16 and 5.22] and appeared originally in [21], [22], and [23] in slightly different form. The forms above can be derived by taking the limit $q \to 1$ in [11] or by repeated application of the addition formula for sine. We will also make frequent use of (2.1.1) and (1.3.5).

4.6.1 AAA Relations

If we divide both sides of the equation in Theorem 4.1.2 by $\Gamma[b,c,d,e,f,g,h,b+c-a,d+c-a,e+c-a,f+c-a,g+c-a,h+c-a,d+b-a,e+b-a,f+b-a,g+b-a,h+b-a]$, we may restate Theorem 4.1.2 in the following form:

Proposition 4.6.1. *We have the AAA relation*

$$\frac{\sin\pi(b-a)}{\Gamma[c-a+(d,e,f,g,h)]}J_{(1,2)}(\vec{x}) + \frac{\sin\pi(a-c)}{\Gamma[b-a+(d,e,f,g,h)]}J_{(1,3)}(\vec{x}) + \frac{\sin\pi(c-b)}{\Gamma[d,e,f,g,h]}J_{(2,3)}(\vec{x}) = 0. \tag{4.6.3}$$

If we define

$$\alpha(\vec{x}) = \frac{\sin\pi(b-a)}{\Gamma[d+c-a,e+c-a,f+c-a,g+c-a,h+c-a]}$$

where $\vec{x} = (a,b,c,d,e,f,g,h)^T \in V$ and recall that $\Phi((123)) = E_{(23)}BE_{(23)}$ and $\Phi((132)) = B$, then this relation may be written in the more symmetric form

$$\alpha(\vec{x})J_{(1,2)}(\vec{x}) + \alpha(\Phi((123))\vec{x})J_{(1,3)}(\vec{x}) + \alpha(\Phi((132))\vec{x})J_{(2,3)}(\vec{x}) = 0,$$

or in other words,

$$\sum_{j=0}^{2}\alpha(\Phi((123)^j)\vec{x})J(\Phi((123)^j)\vec{x}) = 0.$$

Note that the transformation

$$\vec{x} \mapsto E_{(38)} A_3 E_{(48)} \vec{x} = \begin{pmatrix} 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \vec{x}$$

takes $(1,2)$ to itself, $(1,3)$ to $(1,4)$, and $(2,3)$ to $(1,3)$, so that making this substitution in (4.6.3) yields the AAA relation

$$\begin{aligned} &\frac{\sin \pi(c-d) J_{(1,2)}(\vec{x})}{\Gamma[c+d-a, a-b-e+1, a-b-f+1, a-b-g+1, a-b-h+1]} \\ &+\frac{\sin \pi(b-c) J_{(1,4)}(\vec{x})}{\Gamma[b+c-a, a-d-e+1, a-d-f+1, a-d-g+1, a-d-h+1]} \\ &+\frac{\sin \pi(d-b) J_{(1,3)}(\vec{x})}{\Gamma[b+d-a, a-c-e+1, a-c-f+1, a-c-g+1, a-c-h+1]} = 0 \end{aligned} \tag{4.6.4}$$

upon application of the hyperplane relation (2.1.1).

4.6.2 *AAB* Relations

Proposition 4.6.2. *We have the AAB relation*

$$\begin{aligned} &\frac{\sin \pi[b-a, d-a, a-c-(e,f,g,h)] - \sin \pi[b-c, d-c, e, f, g, h]}{\Gamma[c+d-a]} J_{(1,2)}(\vec{x}) \\ &+\frac{\pi^4 \sin \pi[a-c, d-a]}{\Gamma[b+d-a, b-a+(e,f,g,h), 1+a-c-(e,f,g,h)]} J_{(1,3)}(\vec{x}) \\ &+\frac{\pi^4 \sin \pi[a-c, b-c]}{\Gamma[c, e, f, g, h, 1+a-d-(e,f,g,h)]} J_{(2,4)}(\vec{x}) = 0. \end{aligned} \tag{4.6.5}$$

Proof. Into (4.6.3), we substitute

$$\vec{x} \mapsto E_{(34)} \vec{x} = (a, b, d, c, e, f, g, h)^T.$$

We check that this transformation takes $(1,2)$ to itself, $(1,3)$ to $(1,4)$, and $(2,3)$ to $(2,4)$. Thus, (4.6.3) yields

$$\begin{aligned}&\frac{\sin\pi(b-a)J_{(1,2)}(\vec{x})}{\Gamma[c+d-a,e+d-a,f+d-a,g+d-a,h+d-a]}\\&+\frac{\sin\pi(a-d)J_{(1,4)}(\vec{x})}{\Gamma[b+c-a,e+b-a,f+b-a,g+b-a,h+b-a]}\\&+\frac{\sin\pi(d-b)J_{(2,4)}(\vec{x})}{\Gamma[c,e,f,g,h]}=0.\end{aligned}\tag{4.6.6}$$

We multiply (4.6.6) by

$$\frac{\pi^4\sin\pi(b-c)}{\sin\pi(d-b)\Gamma[1+a-d-(e,f,g,h)]},$$

multiply (4.6.4) by

$$\frac{\pi^4\sin\pi(a-d)}{\sin\pi(d-b)\Gamma[b-a+(e,f,g,h)]},$$

and subtract. Simplifying by applying (1.3.5), (2.1.1), and (2.1.6) to each of the resulting coefficients, we obtain the relation

$$\begin{aligned}&\frac{\sin\pi[b-a,b-c,d-a+(e,f,g,h)]-\sin\pi[d-a,d-c,b-a+(e,f,g,h)]}{\sin\pi(d-b)\Gamma[c+d-a]}J_{(1,2)}(\vec{x})\\&+\frac{\pi^4\sin\pi(d-a)}{\Gamma[b+d-a,b-a+(e,f,g,h),1+a-c-(e,f,g,h)]}J_{(1,3)}(\vec{x})\\&+\frac{\pi^4\sin\pi(b-c)}{\Gamma[c,e,f,g,h,1+a-d-(e,f,g,h)]}J_{(2,4)}(\vec{x})=0.\end{aligned}\tag{4.6.7}$$

By (4.6.2),

$$\begin{aligned}&\sin\pi[d-a+(e,f,g,h),b-a,b-c,a-c]\\&=\sin\pi[b-a,a-c-(e,f,g,h),d-b,d-a]\\&\quad-\sin\pi[b-c,e,f,g,h,d-b,d-c]\\&\quad+\sin\pi[a-c,b-a+(e,f,g,h),d-a,d-c],\end{aligned}$$

and so, by multiplying (4.6.7) by $\sin\pi(a-c)$, we obtain the desired relation

$$\begin{aligned}&\frac{\sin\pi[b-a,d-a,a-c-(e,f,g,h)]-\sin\pi[b-c,d-c,e,f,g,h]}{\Gamma[c+d-a]}J_{(1,2)}(\vec{x})\\&+\frac{\pi^4\sin\pi[a-c,d-a]}{\Gamma[b+d-a,b-a+(e,f,g,h),1+a-c-(e,f,g,h)]}J_{(1,3)}(\vec{x})\\&+\frac{\pi^4\sin\pi[a-c,b-c]}{\Gamma[c,e,f,g,h,1+a-d-(e,f,g,h)]}J_{(2,4)}(\vec{x})=0.\end{aligned}$$

□

4.6.3 *ABB* Relations

Proposition 4.6.3. *We have the ABB relation*

$$\begin{aligned}
&\frac{\sin\pi(2b-a)}{\Gamma[1+a-b-(d,e,f,g,h)]}\\
&\cdot(\sin\pi[b,c-(d,e,f,g,h)]+\sin\pi[a-b-2c,a-c-(d,e,f,g,h)])\,J_{(1,2)}(\vec{x})\\
-&\frac{\sin\pi(2c-a)}{\Gamma[1+a-c-(d,e,f,g,h)]}\\
&\cdot(\sin\pi[c,b-(d,e,f,g,h)]+\sin\pi[a-2b-c,a-b-(d,e,f,g,h)])\,J_{(1,3)}(\vec{x})\\
-&\frac{\pi^4\sin\pi[b-c,2b-a,2c-a]}{\Gamma[b,c,b+c-a,1+a-((d,e,f,g,h))]}J_{(2,3)^*}(\vec{x})=0.
\end{aligned}\tag{4.6.8}$$

Proof. Into (4.6.7), we substitute

$$\vec{x}\mapsto E_{(345)}A_4E_{(234)}\vec{x}=\begin{pmatrix}
-1/2 & -1/2 & -1/2 & 3/2 & -1/2 & 1/2 & 1/2 & 1/2\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
-1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2\\
-1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2\\
-1/2 & -1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2\\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
-1 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}\vec{x}.$$

We check that this transformation takes $(1,2)$ to $(1,4)$, $(1,3)$ to $(2,3)^*$, and $(2,4)$ to $(1,2)$. Thus, (4.6.7) yields

$$\begin{aligned}
&\frac{1}{\Gamma[1+a-c-d]\sin\pi(a-c-d-e)}\\
&\cdot(\sin\pi[a-b-c-d,a-b-c-e,d-a+e,b-a+(f,g,h)]\\
&+\sin\pi[c,b-d,b-e,a-((f,g,h))])J_{(1,4)}(\vec{x})\\
-&\frac{\pi^4\sin\pi(b-d)J_{(2,3)^*}(\vec{x})}{\Gamma[b,c,b+d-a,1+a-e-(f,g,h),1+a-((f,g,h))]}\\
-&\frac{\pi^4\sin\pi(a-b-c-d)J_{(1,2)}(\vec{x})}{\Gamma[1+a-b-c,1+a-b-(e,f,g,h),d-a+(e,f,g,h)]}=0.
\end{aligned}\tag{4.6.9}$$

We multiply (4.6.4) by

$$
\begin{aligned}
&-\frac{1}{\Gamma[1+a-c-d]} \\
&\cdot(\sin\pi[a-b-c-d, a-b-c-e, d-a+e, b-a+(f,g,h)] \\
&\quad+\sin\pi[c, b-d, b-e, a-((f,g,h))]),
\end{aligned}
$$

multiply (4.6.9) by

$$
\frac{\sin\pi[b-c, a-c-d-e]}{\Gamma[b+c-a, 1+a-d-(e,f,g,h)]},
$$

and subtract. Simplifying as in Theorem 4.6.2, we obtain the relation

$$
\begin{aligned}
&\frac{1}{\pi\Gamma[1+a-b-(e,f,g,h)]} \\
&\cdot(\sin\pi[b-c, a-b-c, a-b-c-d, a-c-d-e, a-d-(e,f,g,h)] \\
&\quad+\sin\pi[c, b-d, b-e, c-d, a-c-d, a-((f,g,h))] \\
&\quad+\sin\pi[c-d, a-c-d, a-d-e, a-b-(c+d, c+e, f, g, h)])\, J_{(1,2)}(\vec{x}) \\
-&\frac{\sin\pi(d-b)}{\Gamma[b+d-a, 1+a-c-(d,e,f,g,h)]} \\
&\cdot(\sin\pi[a-d-e, a-b-c-d, a-b-c-e, a-b-(f,g,h)] \\
&\quad+\sin\pi[c, b-d, b-e, a-((f,g,h))])\, J_{(1,3)}(\vec{x}) \\
+&\frac{\pi^4\sin\pi[b-c, b-d, a-c-d-e]}{\Gamma[b, c, b+c-a, b+d-a, 1+a-((d,e,f,g,h))]} J_{(2,3)^*}(\vec{x}) = 0.
\end{aligned} \tag{4.6.10}
$$

By (4.6.1), we have

$$
\begin{aligned}
&\sin\pi[c, b-d, b-e, f+g+h-a-b] = \\
&\qquad\qquad \sin\pi[a-c-d-e, a-b-e, a-b-d, 2b+c-a] \\
&\qquad\qquad -\sin\pi[a-d-e, a-b-c-e, a-b-c-d, 2b-a]
\end{aligned}
$$

and so applying (4.6.1) to $\sin\pi[a-f-g, a-f-h, a-g-h, a-2b]$, we have

$$
\begin{aligned}
&\sin\pi[a-2b, c, b-d, b-e, a-((f,g,h))] \\
&= -\sin\pi[c, b-d, b-e, f+g+h-a-b, a-b-f, a-b-g, a-b-h] \\
&\quad -\sin\pi[c, b-d, b-e, b-f, b-g, b-h, c+d+e-a] \\
&= \sin\pi[c+d+e-a, 2b+c-a, a-b-(d,e,f,g,h)] \\
&\quad -\sin\pi[a-2b, a-d-e, a-b-c-d, a-b-c-e, a-b-(f,g,h)] \\
&\quad -\sin\pi[c, c+d+e-a, b-(d,e,f,g,h)]. \qquad (4.6.11)
\end{aligned}
$$

So, if we multiply (4.6.10) by $\sin\pi(a-2b)$, apply (4.6.11) in both the coefficient of $J_{(1,2)}(\vec{x})$ and $J_{(1,3)}(\vec{x})$, and then divide by $\sin\pi(c+d+e-a)$, the coefficient of $J_{(1,2)}(\vec{x})$ becomes

$$
\begin{aligned}
&\frac{1}{\pi\Gamma[1+a-b-(e,f,g,h)]} \\
&\cdot(\sin\pi[b-c, a-b-c, a-b-c-d, 2b-a, a-d-(e,f,g,h)] \\
&\quad +\sin\pi[c-d, a-c-d, 2b+c-a, a-b-(d,e,f,g,h)] \\
&\quad -\sin\pi[c, c-d, a-c-d, b-(d,e,f,g,h)])
\end{aligned}
$$

and the coefficient of $J_{(1,3)}(\vec{x})$ becomes

$$
\begin{aligned}
&-\frac{\sin\pi(d-b)}{\Gamma[b+d-a, 1+a-c-(d,e,f,g,h)]} \\
&\cdot\left(\sin\pi[2b+c-a, a-b-(d,e,f,g,h)] - \sin\pi[c, b-(d,e,f,g,h)]\right).
\end{aligned}
$$

Now, by (4.6.2),

$$
\begin{aligned}
&\sin\pi[b-(e,f,g,h), d-b, c, b+c-d] \\
&= -\sin\pi[b-(e+1,f,g,h), d-b, c, b+c-d] \\
&= \sin\pi[d-b, b+c-a+(e,f,g,h), 2b-a, b+d-a] \\
&\quad -\sin\pi[c, d-a+(e,f,g,h), 2b-a, 2b+c-a] \\
&\quad +\sin\pi[b+c-d, b-a+(e,f,g,h), b-a+d, 2b+c-a], \qquad (4.6.12)
\end{aligned}
$$

so when we multiply (4.6.10) by $\sin\pi(b+c-d)$ and apply (4.6.12), the coefficient of $J_{(1,2)}(\vec{x})$ becomes

$$\frac{\sin\pi(2b-a)}{\pi\Gamma[1+a-b-(e,f,g,h)]}\cdot\Big(\sin\pi[d-a+(e,f,g,h)]\,(\sin\pi[b-c,a-b-c,a-b-c-d,b+c-d]$$
$$-\sin\pi[c,2b+c-a,c-d,a-c-d])+\sin\pi[b-d,c-d,a-b-d,a-c-d,b+c-a+(e,f,g,h)]\Big).$$

We apply (4.6.1) once more:

$$\sin\pi[a-b-c-d,b-c,a-b-c,b+c-d]=$$
$$\sin\pi[b,a-b-d,b-d,a-b-2c]$$
$$+\sin\pi[a-c-d,c,2b+c-a,c-d]$$

and so the coefficient of $J_{(1,2)}(\vec{x})$ becomes

$$\frac{\sin\pi[2b-a,b-d,a-b-d]}{\pi\Gamma[1+a-b-(e,f,g,h)]}$$
$$\cdot\big(\sin\pi[b,a-b-2c,d-a+(e,f,g,h)]+\sin\pi[c-d,a-c-d,b+c-a+(e,f,g,h)]\big).$$

To make the symmetry in the variables d, e, f, g, and h clear, we apply (4.6.2) once again:

$$\sin\pi[d-a+(e,f,g,h),b,b+2c-a,2c-a]$$
$$=\sin\pi[b,c-(e,f,g,h),d-b-c,d-c]$$
$$-\sin\pi[b+2c-a,a-c-(e,f,g,h),d-b-c,c+d-a]$$
$$+\sin\pi[2c-a,a-b-c-(e,f,g,h),d-c,c+d-a],$$

so that, upon multiplication by $\dfrac{\sin\pi(2c-a)\Gamma(b+d-a)}{\sin\pi[b-d,b+c-d]}$, the relation becomes

$$\frac{\sin\pi(2b-a)}{\Gamma[1+a-b-(d,e,f,g,h)]}$$
$$\cdot\left(\sin\pi[b,c-(d,e,f,g,h)]+\sin\pi[a-b-2c,a-c-(d,e,f,g,h)]\right)J_{(1,2)}(\vec{x})$$
$$-\frac{\sin\pi(2c-a)}{\Gamma[1+a-c-(d,e,f,g,h)]}$$
$$\cdot\left(\sin\pi[c,b-(d,e,f,g,h)]+\sin\pi[a-2b-c,a-b-(d,e,f,g,h)]\right)J_{(1,3)}(\vec{x})$$
$$-\frac{\pi^4\sin\pi[b-c,2b-a,2c-a]}{\Gamma[b,c,b+c-a,1+a-((d,e,f,g,h))]}J_{(2,3)^*}(\vec{x})=0$$

as required. □

4.6.4 BBB Relations

Proposition 4.6.4. *We have the BBB relation*

$$\frac{\sin\pi[a, b-a]}{\Gamma[1-(b,e,f,g,h), d-a+(e,f,g,h)]}\cdot$$
$$(\sin\pi[d, 2c+d-a, b-a+(e,f,g,h)]$$
$$-\ \sin\pi[c-b, a-b-c, c+d-a+(e,f,g,h)])\, J_{(1,2)}(\vec{x})$$
$$+\ \frac{\sin\pi[b-a, c+d-b]}{\Gamma[1+a-c-(d,e,f,g,h), b-a+(e,f,g,h)]}$$
$$\cdot\,(\sin\pi[a-b-c, a-(d,e,f,g,h)] - \sin\pi[b+c, d, e, f, g, h])\, J_{(3,4)}(\vec{x})$$
$$+\ \frac{\sin\pi[a, c+d-b]}{\Gamma[c, d, b+c-a, 1+a-((e,f,g,h))]}$$
$$\cdot\,(\sin\pi[a-c, a-d, b-a+(e,f,g,h)] - \sin\pi[b-c, b-d, e, f, g, h])\, J_{(1,3)^*}(\vec{x}) = 0. \tag{4.6.13}$$

Proof. Into (4.6.5), we substitute

$$\vec{x} \mapsto E_{(234)}A_5\vec{x} = \begin{pmatrix} -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\vec{x}.$$

We check that this transformation takes $(1,2)$ to $(1,4)$, $(1,3)$ to $(1,2)$, and $(2,4)$ to $(3,4)$, so that (4.6.5) yields

$$\frac{1}{\Gamma(c)}\,(\sin\pi[d-a, d-b, a-c-(e,f,g,h)] - \sin\pi[c-a, c-b, a-d-(e,f,g,h)])\, J_{(1,4)}(\vec{x})$$
$$+\frac{\pi^4\sin\pi[c-b, c-d]}{\Gamma[d-a+c, d-a+(e,f,g,h), 1-(e,f,g,h)]}J_{(1,2)}(\vec{x})$$
$$+\frac{\pi^4\sin\pi[c-d, d-a]}{\Gamma[b, b-a+(e,f,g,h), 1+a-c-(e,f,g,h)]}J_{(3,4)}(\vec{x}) = 0. \tag{4.6.14}$$

Likewise, we also substitute into (4.6.5)

$$\vec{x} \mapsto E_{(234)(5678)} C A_3 E_{(243)(5876)} \vec{x} =$$
$$\begin{pmatrix} -3/2 & 1/2 & -1/2 & 3/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -3/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \vec{x}.$$

We check that this transformation takes $(1,2)$ to $(1,4)$, $(1,3)$ to $(1,3)^*$, and $(2,4)$ to $(1,2)$, so that (4.6.5) yields

$$\begin{aligned} &\frac{1}{\Gamma(1+a-c-d)} \left(\sin\pi[f,g,h,a-d-e,a-b-c-d,a-b-c-e]\right. \\ &\quad \left. + \sin\pi[a-d,a-e,a-b-c,a-((f,g,h))]\right) J_{(1,4)}(\vec{x}) \\ &+ \frac{\pi^4 \sin\pi[a-d,2a-b-c-d-e]}{\Gamma[b,d,b-a+c,1+a-((e,f,g,h))]} J_{(1,3)^*}(\vec{x}) \\ &+ \frac{\pi^4 \sin\pi[a-b-c-d,2a-b-c-d-e]}{\Gamma[1-c,1-(e,f,g,h),d-a+(e,f,g,h)]} J_{(1,2)}(\vec{x}) = 0. \end{aligned} \tag{4.6.15}$$

Then, if we multiply (4.6.14) by

$$\begin{aligned} &\frac{1}{\Gamma(1+a-c-d)} \left(\sin\pi[f,g,h,a-d-e,a-b-c-d,a-b-c-e]\right. \\ &\quad \left. + \sin\pi[a-d,a-e,a-b-c,a-((f,g,h))]\right), \end{aligned}$$

multiply (4.6.15) by

$$\frac{\sin\pi[d-a,d-b,a-c-(e,f,g,h)] - \sin\pi[c-a,c-b,a-d-(e,f,g,h)]}{\Gamma(c)},$$

and subtract, then we obtain the relation

$$\frac{1}{\Gamma[1-(e,f,g,h),d-a+(e,f,g,h)]}\cdot$$

$$(\sin\pi[c-b,c-d,a-c-d,a-d-e,a-b-c-d,a-b-c-e,f,g,h]$$

$$+\sin\pi[c-b,c-d,a-d,a-e,a-b-c,a-c-d,a-((f,g,h))]$$

$$+\sin\pi[c,d-a,d-b,a-b-c-d,2a-b-c-d-e,a-c-(e,f,g,h)]$$

$$-\sin\pi[c,c-a,c-b,a-b-c-d,2a-b-c-d-e,a-d-(e,f,g,h)])\,J_{(1,2)}(\vec{x})$$

$$-\frac{\pi\sin\pi[c-d,d-a]}{\Gamma[b,1+a-c-(d,e,f,g,h),b-a+(e,f,g,h)]}$$

$$\cdot(\sin\pi[a-d-e,a-b-c-d,a-b-c-e,f,g,h]$$

$$+\sin\pi[a-d,a-e,a-b-c,a-((f,g,h))])\,J_{(3,4)}(\vec{x})$$

$$+\frac{\pi\sin\pi[a-d,2a-b-c-d-e]}{\Gamma[b,c,d,b+c-a,1+a-((e,f,g,h))]}$$

$$\cdot(\sin\pi[d-a,d-b,a-c-(e,f,g,h)]$$

$$-\sin\pi[c-a,c-b,a-d-(e,f,g,h)])\,J_{(1,3)^*}(\vec{x})=0. \tag{4.6.16}$$

Now, by (4.6.1),

$$\sin\pi[a-d,a-e,a-b-c,2a-f-g-h]=$$

$$\sin\pi[d,e,b+c,f+g+h-a]$$

$$-\sin\pi[a,d+e-a,b+c+d-a,a-b-c-e]$$

and so applying (4.6.1) to $\sin\pi[a-f-g,a-f-h,a-g-h,a]$, we see that

$$\sin\pi[a-d,a-e,a-b-c,a-((f,g,h)),a]$$

$$=\sin\pi[a-b-c,a-f-g-h,a-(d,e,f,g,h)]$$

$$-\sin\pi[a-d,a-e,a-b-c,2a-f-g-h,f,g,h]$$

$$=\sin\pi[a-b-c,b+c+d+e-2a,a-(d,e,f,g,h)]$$

$$+\sin\pi[b+c,2a-b-c-d-e,d,e,f,g,h]$$

$$-\sin\pi[a-d-e,a-b-c-d,a-b-c-e,f,g,h,a]. \tag{4.6.17}$$

Also, by (4.6.2),

$$\sin\pi[a-c-(e,f,g,h),a-d,b-d,b-a]$$
$$=-\sin\pi[a-c-(e+1,f,g,h),a-d,b-d,b-a]$$
$$=\sin\pi[a-d,b-a+(e,f,g,h),d-c,a-c]$$
$$-\sin\pi[b-d,e,f,g,h,d-c,b-c]$$
$$+\sin\pi[b-a,d-a+(e,f,g,h),a-c,b-c]. \tag{4.6.18}$$

Multiplying (4.6.16) by $\frac{\sin\pi[a,b-a]}{\Gamma(1-b)\sin\pi[2a-b-c-d-e,d-c]}$, substituting (4.6.17) into the coefficients of $J_{(1,2)}(\vec{x})$ and $J_{(3,4)}(\vec{x})$, and substituting (4.6.18) into the coefficients of $J_{(1,2)}(\vec{x})$ and $J_{(1,3)^*}(\vec{x})$, the relation becomes

$$\frac{1}{\Gamma[1-(b,e,f,g,h),d-a+(e,f,g,h)]}\cdot$$
$$(\sin\pi[b-a,c-b,a-b-c,a-c-d,a-(d,e,f,g,h)]$$
$$-\sin\pi[b-a,b+c,c-b,a-c-d,d,e,f,g,h]$$
$$+\sin\pi[a,c,a-c,a-d,a-b-c-d,b-a+(e,f,g,h)]$$
$$-\ \sin\pi[a,c,b-c,b-d,a-b-c-d,e,f,g,h])\,J_{(1,2)}(\vec{x})$$
$$+\frac{\sin\pi[b,a-d,b-a]}{\Gamma[1+a-c-(d,e,f,g,h),b-a+(e,f,g,h)]}$$
$$\cdot\big(\sin\pi[a-b-c,a-(d,e,f,g,h)]-\sin\pi[b+c,d,e,f,g,h]\big)J_{(3,4)}(\vec{x})$$
$$+\frac{\sin\pi[a,b,a-d]}{\Gamma[c,d,b+c-a,1+a-((e,f,g,h))]}$$
$$\cdot\big(\sin\pi[a-c,a-d,b-a+(e,f,g,h)]-\sin\pi[b-c,b-d,e,f,g,h]\big)J_{(1,3)^*}(\vec{x})=0. \tag{4.6.19}$$

Now, by (4.6.1),

$$\sin\pi[d,b-a,b+c,a-c-d]=$$
$$\sin\pi[b-d,a,c,a-b-c-d]+\sin\pi[b,a-d,c+d,b+c-a],$$

and substituting this into the coefficient of $J_{(1,2)}(\vec{x})$ in (4.6.19) simplifies the coefficient to

$$\frac{\sin\pi(a-d)}{\Gamma[1-(b,e,f,g,h),d-a+(e,f,g,h)]}\cdot$$

$$(\sin\pi[b-a,c-b,a-b-c,a-c-d,a-(e,f,g,h)]$$

$$+\sin\pi[b,b-c,c+d,b+c-a,e,f,g,h]$$

$$+\sin\pi[a,c,a-c,a-b-c-d,b-a+(e,f,g,h)]). \qquad (4.6.20)$$

Now, by (4.6.2),

$$\sin\pi[a-(e,f,g,h),b-a,a-c-d,c+d-b]$$

$$=\sin\pi[a-(e+1,f,g,h),b-a,c+d-a,c+d-b]$$

$$=-\sin\pi[b-a,c+d-a+(e,f,g,h),a,b]$$

$$+\sin\pi[c+d-a,b-a+(e,f,g,h),a,c+d]$$

$$-\sin\pi[c+d-b,e,f,g,h,b,c+d],$$

and if we multiply (4.6.19) by $\frac{\sin\pi(c+d-b)}{\sin\pi(a-d)}$ and substitute this into the coefficient of $J_{(1,2)}(\vec{x})$, then the coefficient becomes

$$\frac{\sin\pi(a)}{\Gamma[1-(b,e,f,g,h),d-a+(e,f,g,h)]}\cdot$$

$$(\sin\pi[c-b,c+d,a-b-c,c+d-a,b-a+(e,f,g,h)]$$

$$-\sin\pi[b,b-a,c-b,a-b-c,c+d-a+(e,f,g,h)]$$

$$+\sin\pi[c,a-c,c+d-b,a-b-c-d,b-a+(e,f,g,h)]).$$

Finally, by (4.6.1),

$$\sin\pi[c-b,c+d,a-b-c,c+d-a]$$

$$=\sin\pi[d,b,2c+d-a,b-a]-\sin\pi[c+d-b,c,a-b-c-d,a-c]$$

and substituting this into the coefficient of $J_{(1,2)}(\vec{x})$ and dividing by $\sin \pi b$ gives the relation

$$\frac{\sin \pi[a, b-a]}{\Gamma[1-(b,e,f,g,h), d-a+(e,f,g,h)]} \cdot$$
$$(\sin \pi[d, 2c+d-a, b-a+(e,f,g,h)]$$
$$- \sin \pi[c-b, a-b-c, c+d-a+(e,f,g,h)])\, J_{(1,2)}(\vec{x})$$
$$+ \frac{\sin \pi[b-a, c+d-b]}{\Gamma[1+a-c-(d,e,f,g,h), b-a+(e,f,g,h)]}$$
$$\cdot (\sin \pi[a-b-c, a-(d,e,f,g,h)] - \sin \pi[b+c, d, e, f, g, h])\, J_{(3,4)}(\vec{x})$$
$$+ \frac{\sin \pi[a, c+d-b]}{\Gamma[c, d, b+c-a, 1+a-((e,f,g,h))]}$$
$$\cdot (\sin \pi[a-c, a-d, b-a+(e,f,g,h)] - \sin \pi[b-c, b-d, e, f, g, h])\, J_{(1,3)^*}(\vec{x}) = 0,$$

as required. □

4.6.5 *ABC* Relations

Proposition 4.6.5. *We have the ABC relation*

$$(\sin \pi[b, a-c, a-b-c, 2b-a, c-(d,e,f,g,h), a-b-(d,e,f,g,h)]$$
$$- \sin \pi[c, a-b, a-b-c, 2c-a, b-(d,e,f,g,h), a-c-(d,e,f,g,h)]$$
$$- \sin \pi[b, c, b-c, 2a-2b-2c, a-b-(d,e,f,g,h), a-c-(d,e,f,g,h)])\, J_{(1,2)}(\vec{x})$$
$$+ \frac{\pi^5 \sin \pi[b-c, 2c-a, a-b-c]}{\Gamma[d, e, f, g, h, 1+a-c-(d,e,f,g,h)]}$$
$$\cdot (\sin \pi[a-2b-c, a-b-(d,e,f,g,h)] + \sin \pi[c, b-(d,e,f,g,h)])\, J_{(2,3)}(\vec{x})$$
$$+ \frac{\pi^9 \sin \pi[2b-a, 2c-a, a-c, b-c, a-b-c]}{\Gamma[b, c, b+c-a, b-a+(d,e,f,g,h), 1+a-((d,e,f,g,h))]} J_{(2,3)^*}(\vec{x}) = 0. \qquad (4.6.21)$$

Proof. Multiplying (4.6.3) by

$$-\frac{\sin \pi(2c-a)}{\Gamma[1+a-c-(d,e,f,g,h)]}$$
$$\cdot (\sin \pi[c, b-(d,e,f,g,h)] + \sin \pi[a-2b-c, a-b-(d,e,f,g,h)])$$

and (4.6.8) by

$$\frac{\sin \pi(a-c)}{\Gamma[b-a+(d,e,f,g,h)]}$$

and subtracting, we have after some trivial simplifications similar to the above that

$$
\begin{aligned}
&(\sin\pi[a-b, 2c-a, 2b+c-a, a-b-(d,e,f,g,h), a-c-(d,e,f,g,h)] \\
&\quad -\sin\pi[a-b, 2c-a, c, b-(d,e,f,g,h), a-c-(d,e,f,g,h)] \\
&\quad +\sin\pi[a-c, 2b-a, a-b-2c, a-b-(d,e,f,g,h), a-c-(d,e,f,g,h)] \\
&\quad +\sin\pi[a-c, 2b-a, b, c-(d,e,f,g,h), a-b-(d,e,f,g,h)])\, J_{(1,2)}(\vec{x}) \\
&+\frac{\pi^5\sin\pi[b-c, 2c-a]}{\Gamma[d,e,f,g,h,1+a-c-(d,e,f,g,h)]} \\
&\quad \cdot(\sin\pi[a-2b-c, a-b-(d,e,f,g,h)]+\sin\pi[c, b-(d,e,f,g,h)])\, J_{(2,3)}(\vec{x}) \\
&+\frac{\pi^9\sin\pi[2b-a, 2c-a, a-c, b-c]}{\Gamma[b,c,b+c-a,b-a+(d,e,f,g,h),1+a-((d,e,f,g,h))]} J_{(2,3)^*}(\vec{x}) = 0. \qquad (4.6.22)
\end{aligned}
$$

Now, by (4.6.1),

$$
\begin{aligned}
&\sin\pi[2c-a, a-b, 2b+c-a, a-b-c] \\
&= \sin\pi[a-c, b+c-a, a-2b, b+2c-a] - \sin\pi[c, b-c, 2a-2b-2c, b], \qquad (4.6.23)
\end{aligned}
$$

so if we multiply (4.6.22) by $\sin\pi(a-b-c)$ and apply (4.6.23) in the first term of the coefficient of $J_{(1,2)}(\vec{x})$, the relation (4.6.22) simplifies to

$$
\begin{aligned}
&(\sin\pi[b, a-c, a-b-c, 2b-a, c-(d,e,f,g,h), a-b-(d,e,f,g,h)] \\
&\quad -\sin\pi[c, a-b, a-b-c, 2c-a, b-(d,e,f,g,h), a-c-(d,e,f,g,h)] \\
&\quad -\sin\pi[b, c, b-c, 2a-2b-2c, a-b-(d,e,f,g,h), a-c-(d,e,f,g,h)])\, J_{(1,2)}(\vec{x}) \\
&+\frac{\pi^5\sin\pi[b-c, 2c-a, a-b-c]}{\Gamma[d,e,f,g,h,1+a-c-(d,e,f,g,h)]} \\
&\quad \cdot(\sin\pi[a-2b-c, a-b-(d,e,f,g,h)]+\sin\pi[c, b-(d,e,f,g,h)])\, J_{(2,3)}(\vec{x}) \\
&+\frac{\pi^9\sin\pi[2b-a, 2c-a, a-c, b-c, a-b-c]}{\Gamma[b,c,b+c-a,b-a+(d,e,f,g,h),1+a-((d,e,f,g,h))]} J_{(2,3)^*}(\vec{x}) = 0
\end{aligned}
$$

as required. □

4.6.6 Conclusion

Combining the above subsections, we obtain our main result of this section.

Theorem 4.6.6. *(i) Let μ_1, μ_2, and μ_3 be elements of M_J such that no two of the μ_j's are in the same right coset of G_J in M_J. Then there is a relation of the form*

$$\gamma_1 J_{\mu_1}(\vec{x}) + \gamma_2 J_{\mu_2}(\vec{x}) + \gamma_3 J_{\mu_3}(\vec{x}) = 0, \tag{4.6.24}$$

where γ_1, γ_2, and γ_3 are entire, rational combinations of gamma and sine functions, whose arguments are $\mathbb{Z}$-affine combinations of a,b,c,d,e,f,g, and h.

(ii) For any $\ell \in \{1,2,3\}$, pick j and k so that $\{j,k\} = \{1,2,3\} \setminus \{\ell\}$. Then, in a relation of the form (4.6.24), *each coefficient γ_ℓ may be written as a sum of n monomials in gamma and sine functions, where $\sqrt{32n}$ is the Euclidean distance between μ_j and μ_k.*

(iii) If (μ_1,μ_2,μ_3) and (ν_1,ν_2,ν_3) are triples of the same Euclidean type, then a three-term relation among J_{μ_1}, J_{μ_2}, and J_{μ_3} can be transformed into one involving J_{ν_1}, J_{ν_2}, and J_{ν_3} by the application of a single change of variable

$$\vec{x} \mapsto \rho\vec{x} \qquad\qquad (\rho \in M_J)$$

to all elements (including the coefficients) of the first relation.

Proof. The assertions follow by combining Theorem 4.5.8 with Propositions 4.6.1–4.6.5. □

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