Midterm Review for MATH 3510

This exam will cover Sections 2.1–2.3, 4.1–4.2, 5.1–5.6, and 7.1–7.3. You will be expected to know the main terminology, theorems, and distributions that we've covered, to be able to calculate probabilities, expected values, and standard deviations, to prove basic formulas (typically, through computation), and to apply this knowledge in concrete problems. You will be permitted to use a calculator. Features that most calculators cannot handle (for example, non-graphing calculators typically can't do $\Phi(z)$) may be left unsimplified, though some problems may require you to have a "ballpark" idea of how big these numbers are. If you are unfamiliar with some background cultural knowledge on the exam (for example, you don't know what the 52 cards in a standard deck are) or have trouble figuring out how to enter something into your calculator (for example, $\binom{n}{k}$), feel free to ask me during the exam. As my first goal is to make sure that you understand the basic concepts, many problems from the exam will be taken directly from Sections 1–3 below (and, of course, knowing this material is important for many other problems). As such, I suggest that you study them first. Other problems will be at about the difficulty level of the homework problems. In fact, many of the problems in Section 4 below come directly from the problems in the book; if you're looking for more problems, many other book problems are worth considering.

1 Terminology

1. Be able to define: sample space, event, random variable, expected value (of a discrete or continuous random variable), variance, standard deviation, disjoint, mutually exclusive, the standard normal distribution function $\Phi(z)$. (We've also used other terminology that I won't ask you to define but will expect you to know well enough to use, e.g., "independent and identically distributed," "continuous random variable," etc.)

2. Be able to state: the Kolmogorov axioms, the law of large numbers, the central limit theorem, the square-root law (cf. Property 4 in Section 5.3), the continuity properties of probability (cf. Section 7.1.3), the rules 7.1–7.4 in Section 7.3 (you don't need to know the numbers 7.1–7.4 that the book gives these rules, of course).

3. Be able to explain why we care about the law of large numbers and the central limit theorem.

4. For each of the distributions we've talked about (Bernoulli experiment, binomial distribution, Poisson distribution, normal distribution), be able to give its probability mass function or probability density function (as appropriate), its expected value and standard deviation, and the type of situations in which it's used.

2 Calculation

5. Calculate the expected value and standard deviation for: a Bernoulli distribution (that is, a random variable that equals 1 on the success of a Bernoulli experiment and 0 otherwise), the binomial distribution, and the Poisson distribution. (You should also know these quantities for a normal distribution, but I won't ask you to calculate them.)

6. Be able to calculate the probability mass function, expected value, and standard deviation for discrete random variables X with finitely many mass points, where X is not one of the distributions we've talked about. For example:
   (a) Roll a fair die and let X be the number rolled.
   (b) Roll two fair dice and (i) let X be their sum, (ii) let X be the larger of the numbers rolled, (iii) let X be the absolute difference between the numbers rolled (that is, $X = |x - y|$ if (x, y) are the numbers rolled).
   (c) Roll a fair die and flip a fair coin. If the coin comes up heads, let X be the number rolled. If the coin comes up tails, let X be the negation of the number rolled.

3 Basic Proofs

Unless otherwise stated, you may use the Kolmogorov axioms and any other results that we've seen. In particular, you may assume without proof that $E(aX + bY) = aE(X) + bE(y)$ and that $E(aX + b) = aE(X) + b$. If you use a rule that requires a certain hypothesis (such as two events being disjoint), be sure to explain why the hypothesis is satisfied.

7. Be able to prove Rules 7.1–7.3 in Section 7.3, using only the Kolmogorov axioms.
8. Let A and B be events where the set A is a subset of the set B. Show that $P(A) \leq P(B)$.
9. For any constants a and b and any random variable X, show that $\sigma^2(aX + b) = a^2\sigma^2(X)$.
10. Recall that the covariance of random variables X and Y is defined to be $\text{cov}(X, Y) = E[(X - E(X))(Y - E(Y))]$ and that the correlation coefficient of X and Y is defined to be $\rho(X, Y) = \frac{\text{cov}(X,Y)}{\sigma(X)\sigma(Y)}$. Suppose that $Y = aX + b$, where $a \neq 0$. Show that $\rho(X, Y)$ is either 1 or -1.
11. For any random variables X and Y, show that $\sigma^2(X + Y) = \sigma^2(X) + \sigma^2(Y) + 2\,\text{cov}(X, Y)$, where $\text{cov}(X, Y)$ is defined in the previous problem.
12. Show that $\text{var}(X) = E(X^2) - E(X)^2$.
13. Let X be a continuous random variable. Show that $P(a \leq X \leq b) = P(a < X < b)$ for any real numbers a and b with $a < b$.

4 Example Problems

14. The life of the battery of a certain laptop (between charges) is normally distributed with a mean of 20 hours and a standard deviation of 3 hours. What is the probability that the battery of the laptop will last for between 14 and 23 hours?
15. The event A has a probability of $\frac{2}{3}$ and there is a probability of $\frac{3}{4}$ that at least one of the events A and B occurs. What are the smallest and largest possible values for the probability of event B?

16. A military early-warning installation is constructed in a desert. The installation consists of five main detectors and a number of reserve detectors. If fewer than five detectors are working, the installation ceases to function. Every two months an inspection of the installation is mounted and at that time all detectors are replaced by new ones. There is a probability of 0.05 that any given detector will cease to function during the period between inspections. The detectors function independently of one another. How many reserve detectors are needed to ensure a probability of less than 0.1% that the system will cease to function between inspections?

17. Jean claims to have flipped a fair coin 1000 times and gotten 427 heads. Let X be the number of heads out of 1000 flips. Explain why X is approximately normal and calculate the probability of getting 427 or fewer heads out of 1000 flips. Statistically speaking, do you have reason to doubt Jean's claim?

18. Three people each write down the numbers $1, \ldots, 10$ in a random order. Calculate a Poisson approximation for the probability that the three people all have at least one number in the same position.

19. Each year in Houndsville, an average of 81 letter carriers are bitten by dogs. In the past year, 117 such incidents were reported. Is this number exceptionally high?

20. Mary and Norman have a joint checking account with \$1,000 in it. If Mary writes a check for between \$0 and \$$m$ dollars with $0 < m < 1000$ and with each amount equally likely and Norman writes a check for between \$0 and \$$n$ dollars with $0 < n < 1000$ and with each amount equally likely, what is the probability that Mary and Norman will overdraw their account?

21. The keeper of a certain king's treasure receives the task of filling each of 100 urns with 100 gold coins. While fulfilling this task, he substitutes one lead coin for one gold coin in each urn. The king suspects deceit on the part of the sentry and has two methods at his disposal of auditing the contents of the urns. The first method consists of randomly choosing one coin from each of the 100 urns. The second method consists of randomly choosing four coins from each one of 25 of the 100 urns. Which method provides the largest probability of uncovering the deceit?

22. A dart is thrown at random on a rectangular board. The board measures 20 cm by 50 cm. A hit occurs if the dart lands within 5 cm of any of the four corner points of the board. What is the probability of a hit?

23. Calculate a Poisson approximation for the probability that two consecutive numbers will appear in a lotto drawing of six numbers from the numbers $1, \ldots, 45$.

24. An integer is chosen at random from the numbers $1, \ldots, 1000$. What is the probability that the chosen integer is divisible by at least one of 2, 3, and 5. What is the probability that the chosen integer is divisible by none of 2, 3, and 5?