1. (16 pts) Find all matrices that commute with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Write $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Setting $AB = BA$ and equating the four components of the product matrices, we see that $c = 0$ and $a = d$, so all matrices of the form $B = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ where a and b are any real numbers (and only matrices of this form) will commute with A.

2. (16 pts) Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 2 & 2 & -1 \end{bmatrix}$ and $\mathfrak{B} = \{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \}$ be a basis of $\mathbb{R}^3$.

Find the $\mathfrak{B}$-matrix of the linear transformation $T(\vec{x}) = A\vec{x}$.

The easiest method is probably to compute this by columns: $[T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and doing the same thing with the other basis elements shows that $[T(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix})]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $[T(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix})]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so that the $\mathfrak{B}$-matrix of T is $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Using the method $B = S^{-1}AS$ to compute this is also not too ugly in this case, since the inverse of S is fairly nice. If you did it this way, you should have ended up with $S^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

3. (16 pts) Let A be an $n \times m$ matrix and V be a subspace of $\mathbb{R}^n$. Define

$$W = \{\vec{x} \in \mathbb{R}^m | A\vec{x} \in V\}.$$

Show that W is a subspace of $\mathbb{R}^m$.

We need to check the three properties in the definition of a subspace of $\mathbb{R}^m$:

(a) Note that $A\vec{0} = \vec{0}$ and $\vec{0} \in V$ since V is a subspace of $\mathbb{R}^n$ and all subspaces contain $\vec{0}$. Thus, $\vec{0} \in W$.

(b) Let $\vec{x}, \vec{y} \in W$. That is, $A\vec{x}, A\vec{y} \in V$. Then $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} \in V$ since subspaces are closed under addition. Thus, $\vec{x} + \vec{y} \in W$.

(c) Let $\vec{x} \in W$, $k \in \mathbb{R}$. That is, $A\vec{x} \in V$. Then $A(k\vec{x}) = kA\vec{x} \in V$ since subspaces are closed under scalar multiplication. Thus, $k\vec{x} \in W$.

The three properties hold, so W is a subspace of $\mathbb{R}^m$.

Aside: We can use the above to show that the solutions to (for example)

$$ax + by + cz = dx + ey + fz = gx + hy + jz$$

form a subspace of $\mathbb{R}^3$. Note that the solutions to these equalities satisfy $A\vec{x} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for some $k \in \mathbb{R}$ where $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$, so if we let $V = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix})$, then the set of solutions is precisely the set W that we proved was a subspace above.

A few special cases of the above: 1) If $\text{im}(A) \subseteq V$, then $W = \mathbb{R}^m$. 2) If $V = \{\vec{0}\}$, then $W = \ker(A)$.

4. Consider the matrix

$$A = \begin{bmatrix} 2 & -2 & -3 & -3 & 5 \\ 4 & -4 & -3 & -2 & 1 \\ 1 & -1 & -3 & -3 & 7 \\ 1 & -1 & 2 & 2 & -8 \end{bmatrix} \text{ with } \operatorname{rref}(A) = \begin{bmatrix} 1 & -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (8 pts) Find a basis of $\operatorname{im}(A)$. What is the dimension of $\operatorname{im}(A)$?

We know that the columns of A corresponding to the columns of $\operatorname{rref}(A)$ with leading 1's will form a bais of $\operatorname{im}(A)$, so that $\left\{ \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -3 \\ 2 \end{bmatrix} \right\}$ is a basis of $\operatorname{im}(A)$, so $\dim(\operatorname{im}(A)) = 3$.

(b) (8 pts) Find a basis of $\ker(A)$. What is the dimension of $\ker(A)$?

From $\operatorname{rref}(A)\vec{x} = \vec{0}$, we see that the solutions to $A\vec{x} = \vec{0}$ are vectors of the form

$$\vec{x} = \begin{bmatrix} s + 2t \\ s \\ 3t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix},$$

so that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\ker(A)$, so $\dim(\ker(A)) = 2$.

Note that $\dim(\operatorname{im}(A)) + \dim(\ker(A)) = 5$, as guaranteed by the Rank-Nullity Theorem.

5. (12 pts) Let A be an $n \times m$ matrix. Let $\vec{v_1}, \ldots, \vec{v_k}$ be a basis of $\ker(A)$ and $\vec{w_1}, \ldots, \vec{w_\ell}$ be a basis of $\operatorname{im}(A)$. For $i = 1, \ldots, \ell$, let $\vec{x_i} \in \mathbb{R}^m$ be a vector such that $A\vec{x_i} = \vec{w_i}$.

   Show that $\mathfrak{B} = \{\vec{v_1}, \ldots, \vec{v_k}, \vec{x_1}, \ldots, \vec{x_\ell}\}$ is a basis of $\mathbb{R}^m$.

   (Hint: It may be useful to determine what $k + \ell$ equals, in terms of n and m.)

   By Rank-Nullity, $k + \ell = \dim(\ker(A)) + \dim(\operatorname{im}(A)) = m$. We know that any m linearly independent vectors in $\mathbb{R}^m$ will form a basis of $\mathbb{R}^m$ (and similarly for m vectors that span $\mathbb{R}^m$), so it is enough to show that the vectors of $\mathfrak{B}$ either are linearly independent or span $\mathbb{R}^m$. Showing linear independence is easier, but I'll show how to do it either way below. Note that you only need to do one of the two below (unless you didn't figure out that $k + \ell = m$, in which case you'd need to show both).

   Let $T(\vec{x}) = A\vec{x}$. (Technically, you don't need to do this: you can use A below everywhere that I use T. I just want to view things in terms of linear transformations instead of matrices for a reason that I'll explain at the end. Note $\operatorname{im}(T) = \operatorname{im}(A)$ and $\ker(T) = \ker(A)$ by definition.)

   Linear Independence: Consider the relation

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} + d_1\vec{x_1} + \cdots + d_\ell\vec{x_\ell} = \vec{0}. \tag{1}$$

   We want to show that only the trivial relation exists; that is, that $c_1 = \cdots = c_k = d_1 = \cdots = d_\ell = 0$.

   Let's apply T to both sides of (1). Since we can split on addition and scalar multiplication (since T is a linear transformation), we get:

$$c_1T(\vec{v_1}) + \cdots + c_kT(\vec{v_k}) + d_1T(\vec{x_1}) + \cdots + d_\ell T(\vec{x_\ell}) = T(\vec{0}).$$

   Note that $T(\vec{v_i}) = \vec{0}$ since $\vec{v_i} \in \ker(T)$ for $i = 1, \ldots k$ and we chose x_i so that $T(\vec{x_i}) = \vec{w_i}$ for $i = 1, \ldots, \ell$. Also, $T(\vec{0}) = \vec{0}$. So the above equation becomes

$$d_1\vec{w_1} + \cdots + d_\ell\vec{w_\ell} = \vec{0}.$$

   But, we know that $\vec{w_1}, \ldots, \vec{w_\ell}$ are linearly independent, so they satisfy only the trivial relation, so $d_1 = \cdots = d_\ell = 0$. Plugging these values into (1), we're left with

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0}.$$

   Since we also know that $\vec{v_1}, \ldots, \vec{v_k}$ are linearly independent, they also satisfy only the trivial relation, so $c_1 = \cdots = c_k = 0$. Thus, (1) is the trivial relation, so that the vectors of $\mathfrak{B}$ are linearly independent.

   Spanning Set: Let $\vec{x} \in \mathbb{R}^m$. We will show that $\vec{x} \in \operatorname{span}(\vec{v_1}, \ldots, \vec{v_k}, \vec{x_1}, \ldots, \vec{x_\ell})$ by explicitly computing it as a linear combination of these vectors. First, note that $T(\vec{x}) \in \operatorname{im}(T)$, so there exist unique scalars $d_1, \ldots, d_\ell$ such that $T(\vec{x}) = d_1\vec{w_1} + \cdots + d_\ell\vec{w_\ell}$ since the vectors $\vec{w_1}, \ldots, \vec{w_\ell}$ form a basis of $\operatorname{im}(T)$. Let $\vec{u} = d_1\vec{x_1} + \cdots + d_\ell\vec{x_\ell}$ for the same scalars d_i. Then $T(\vec{u}) = d_1\vec{w_1} + \cdots + d_\ell\vec{w_\ell} = T(\vec{x})$ (again, since we can split on addition and scalar multiplication).

   Then $T(\vec{x} - \vec{u}) = T(\vec{x}) - T(\vec{u}) = \vec{0}$, so $\vec{x} - \vec{u} \in \ker(T)$. Thus, $\vec{x} - \vec{u} = c_1\vec{v_1} + \cdots + c_k\vec{v_k}$ for some scalars $c_1, \ldots, c_k$ since $\vec{v_1}, \ldots, \vec{v_k}$ form a basis of $\ker(T)$. Thus,

$$\vec{x} = c_1\vec{v_1} + \cdots + c_k\vec{v_k} + \vec{u} = c_1\vec{v_1} + \cdots + c_k\vec{v_k} + d_1\vec{x_1} + \cdots + d_\ell\vec{x_\ell},$$

   so that $\vec{x} \in \operatorname{span}(\vec{v_1}, \ldots, \vec{v_k}, \vec{x_1}, \ldots, \vec{x_\ell})$. Since $\vec{x} \in \mathbb{R}^m$ was arbitrary, this shows that the vectors of $\mathfrak{B}$ span $\mathbb{R}^m$.

Aside: Note that this provides a second proof of the Rank-Nullity Theorem. If we didn't know that $k+\ell=m$, we could prove both linear independence and spanning as above, which would show that $\mathfrak{B}$ is a basis of $\mathbb{R}^m$. Since $\mathfrak{B}$ contains $k+\ell$ vectors and we know that every basis of $\mathbb{R}^m$ has m vectors, we now would then see that $\dim(\operatorname{im}(A))+\dim(\ker(A))=k+\ell=m$.

Why did I use T above instead of A? Because this generalizes to abstract vector spaces. In Section 4.2, we'll define linear transformations, images, and kernels in abstract vector spaces. Then, following the same argument above exactly will prove that for any linear transformation T from a finite-dimensional vector space V to a vector space W, $\dim(\operatorname{im}(T))+\dim(\ker(T))=\dim(V)$. Cool. "But where did that 'finite-dimensional' bit come from," I hear you ask. Good question. Above, we've assumed that $\operatorname{im}(T)$ and $\ker(T)$ are finite-dimensional, which will clearly be the case if V is finite-dimensional, but may not be the case if V is infinite-dimensional.

Aside 2: Some people tried to use one linear relation for the vectors $\vec{v_1},\ldots,\vec{v_k}$ and one linear relation for $\vec{x_1},\ldots,\vec{x_\ell}$ instead of combining them into one relation like we did above in (1). This doesn't work. For example, the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}$ are linearly independent and the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ are linearly independent, but the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ are not linearly independent (since $\begin{bmatrix}0\\1\\0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\\2\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}$ is redundant). Linear independence is a property of a *set* of vectors, not of the individual vectors themselves, so you need to check them all together.

6. (4 pts each) Decide whether the following statements are true or false. You do not need to show work.

Let A be an $n \times n$ matrix. If A is similar to I_n, then $A = I_n$.

$\boxed{\text{True}}$ False

If A is similar to I_n, then there is an invertible matrix S such that $A = S^{-1} I_n S = S^{-1} S = I_n$.

Let A be an invertible $n \times n$ matrix such that $A^2 = A$. Then $A = I_n$.

$\boxed{\text{True}}$ False

Since A is invertible, we can multiply both sides by A^{-1}: $A^{-1}A^2 = A^{-1}A$, so $A = I_n$. Note that we saw on the review that this is not necessarily the case if we don't know that A is invertible.

Let A be an $n \times n$ matrix such that $\operatorname{im}(A) = \ker(A)$. Then n must be even.

$\boxed{\text{True}}$ False

By Rank-Nullity, $\dim(\operatorname{im}(A)) = \dim(\ker(A)) = \frac{n}{2}$. If n is odd, this will not be an integer, which is impossible since the dimension must be an integer.

Let A be a 2×2 matrix that represents a reflection across a line through the origin. Then A is similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$\boxed{\text{True}}$ False

Recall that matrices will be similar if they represent the same transformation, but in different bases. Since we know how to construct the $\mathfrak{B}$-matrix of a transformation column-by-column, this question is asking if we can find a basis $\mathfrak{B} = \{\vec{v}, \vec{w}\}$ of $\mathbb{R}^2$ such that $A\vec{v} = \vec{v}$ and $A\vec{w} = -\vec{w}$. One way to do this is to let $\vec{v}$ be a unit vector on the line and $\vec{w}$ be a unit vector perpendicular to the line. From the geometry, it's clear that $A\vec{v} = \vec{v}$ and $A\vec{w} = -\vec{w}$ and since we know that perpendicular unit vectors are linearly independent, $\mathfrak{B}$ really is a basis. Algebraically, this tells us that if we let $S = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$, then S is invertible and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = S^{-1}AS$.

Let A be an $n \times n$ matrix such that the columns of A are linearly independent. Then $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$.

$\boxed{\text{True}}$ False

This is one of the conditions in Summary 3.3.10.

Let A be an $n \times m$ matrix. Then $\dim(\operatorname{im}(A)) + \dim(\ker(A)) = n$.

True $\boxed{\text{False}}$

By the Rank-Nullity Theorem, $\dim(\operatorname{im}(A)) + \dim(\ker(A)) = m$, not n.
