SCHUBERT VARIETIES AND FREE BRAIDEDNESS

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Abstract. We give a simple necessary and sufficient condition for a Schubert variety X_w to be smooth when w is a freely braided element of a simply laced Weyl group; such elements were introduced by the authors in a previous work. This generalizes in one direction a result of Fan concerning varieties indexed by short-braid avoiding elements. We also derive generating functions for the freely braided elements that index smooth Schubert varieties. All results are stated and proved only for the simply laced case.

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Introduction

The "freely braided elements" of a simply laced Coxeter group were defined in [**7**]. The idea behind the definition is that although it may be necessary to use long braid relations in order to pass between two reduced expressions for a freely braided

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element, the necessary long braid relations in a certain sense do not interfere with one another.

In this paper, we study Schubert varieties indexed by freely braided elements of a simply laced Weyl group; the non-simply-laced case will not be considered. Specifically, we are interested in the problem of determining which of these varieties are smooth. The general problem of characterising smoothness for Schubert varieties has been investigated by numerous authors in recent years. For example, Lakshmibai–Sandhya [**12**] have shown that in the type A_n setting, a Schubert variety X_w is nonsingular if and only if the permutation w avoids the patterns 3412 and 4231. Billey [**1**] has obtained similar pattern-avoidance conditions for rational smoothness in types B_n, C_n and D_n. General necessary and sufficient conditions for smoothness have been given by Kumar [**11**], and recently Billey–Postnikov [**3**] have found a criterion for smoothness in terms of patterns in root systems. Important results have also been obtained by Carrell, Peterson and others; for an overview of the subject, the reader is referred to [**2**].

A main source of motivation for the present work is the paper [**6**], in which Fan studies short-braid avoiding elements of an arbitrary Weyl group and determines which of these elements correspond to smooth Schubert varieties. Fan found that for a short-braid avoiding element w, the corresponding Schubert variety X_w is nonsingular if and only if w is a product of $\ell(w)$ distinct generators. Formulae for the enumeration of such elements were also provided. Since in the simply laced setting short-braid avoiding elements are a special case of freely braided elements, it is natural to try to generalize Fan's results to the freely braided case, and this is a principal goal of the present article. Theorem 3.2 provides a particularly simple characterisation of those freely braided w for which X_w is smooth. A by-product of our argument is that we can describe exactly when the deletion of a letter from a reduced expression for a freely braided element results in another (not necessarily freely braided) reduced expression (see Proposition 2.4 and Remark 2.5). The answer here is perhaps the simplest that can be expected, and generalizes in one direction a deletion result of Fan [**6**, Theorem 1] concerning short-braid avoiding elements of Weyl groups. Recently, Hagiwara–Ishikawa–Tagawa [**9**] have generalized the same deletion result in a different way to a wider class of Coxeter groups.

In Theorem 4.2, we present generating functions for the number of smooth varieties indexed by freely braided elements. These may easily be converted to explicit formulae using standard methods.

1. Preliminaries

Let W be a simply laced Coxeter group with set of distinguished generators $S = \{s_i : i \in I\}$ and Coxeter matrix $(m_{ij})_{i,j \in I}$. Denote by I^* the free monoid on I, and let $\phi : I^* \longrightarrow W$ be the surjective morphism of monoid structures satisfying $\phi(i) = s_i$ for all $i \in I$. We say that a word $\mathbf{i} \in I^*$ *represents* its image $w = \phi(\mathbf{i}) \in W$; further, if the length of $\mathbf{i}$ (i.e., the number of factors used to express $\mathbf{i}$ as a product of letters from I) is minimal among the lengths of all the words that represent w, then we call $\mathbf{i}$ a *reduced expression* for w. The length of w, denoted by $\ell(w)$, is then equal to the length of $\mathbf{i}$.

Let V be a real vector space with basis $\{\alpha_i : i \in I\}$, and denote by B the *Coxeter form* on V associated to W. This is the symmetric bilinear form satisfying $B(\alpha_i, \alpha_j) = -\cos \frac{\pi}{m_{ij}}$ for all $i, j \in I$. We view V as the underlying space of a reflection representation of W, determined by the equalities $s_i \alpha_j = \alpha_j - 2B(\alpha_j, \alpha_i)\alpha_i$ for all $i, j \in I$.

Denote by Φ the *root system* of W, i.e., $\{w\alpha_i : w \in W \text{ and } i \in I\}$. Let Φ^+ be the set of all $\beta \in \Phi$ such that β is expressible as a linear combination of the α_i with nonnegative coefficients, and let $\Phi^- = -\Phi^+$. We have $\Phi = \Phi^+ \cup \Phi^-$ (disjoint). The elements of Φ^+ (respectively, Φ^-) are called *positive* (respectively, *negative*) roots. The α_i are also referred to as *simple* roots.

Associated to each $w \in W$ is the *inversion set* $\Phi(w) = \Phi^+ \cap w^{-1}(\Phi^-)$, which has cardinality $\ell(w)$. Given any reduced expression $i_1 i_2 \cdots i_n$ for w, we have $\Phi(w) = \{r_1, r_2, \ldots, r_n\}$, where $r_1 = \alpha_{i_n}$ and $r_l = s_{i_n} \cdots s_{i_{n-l+2}}(\alpha_{i_{n-l+1}})$ for all $l \in \{2, \ldots, n\}$. The sequence $\bar{r} = (r_1, r_2, \ldots, r_n)$ is called the *root sequence* of $i_1 i_2 \cdots i_n$, or a root sequence *for* w. Note that any initial segment of a root sequence is also a root sequence for some element of W.

Let $\mathbf{i}, \mathbf{j} \in I^*$ and let $i, j, k \in I$. We call the substitution $\mathbf{i}ij\mathbf{j} \to \mathbf{i}ji\mathbf{j}$ a *commutation* or *short braid move* if $m_{ij} = 2$, and if $m_{ij} = 3$, we call $\mathbf{i}iji\mathbf{j} \to \mathbf{i}jij\mathbf{j}$ a *long braid move*. Applying a braid move to a reduced expression corresponds to applying a permutation to the root sequence of that reduced expression [**7**, Proposition 3.1.1]. Specifically, let $w \in W$ and suppose that $\mathbf{i}ij\mathbf{j}$ is a reduced expression for w. Let $\bar{r} = (r_l)$ be the associated root sequence, and let n be the length of $\mathbf{j}$. Then $m_{ij} = 2$ if and only if r_{n+1} and r_{n+2} are mutually orthogonal, in which case the root sequence $\bar{r}'$ of $\mathbf{i}ji\mathbf{j}$ can be obtained from $\bar{r}$ by interchanging r_{n+1} and r_{n+2}. Employing again the terminology used above for words, we say that the passage from $\bar{r}$ to $\bar{r}'$ is obtained by a *commutation* or *short braid move*. Suppose now that $\mathbf{i}ijk\mathbf{j}$ is a reduced expression for w, and let $\bar{r} = (r_l)$ be the associated root sequence. Again, denote the length of $\mathbf{j}$ by n. Then $i = k$ if and only if $r_{n+1} + r_{n+3} = r_{n+2}$, in which case the root sequence $\bar{r}'$ of $\mathbf{i}jij\mathbf{j}$ can be obtained from $\bar{r}$ by interchanging r_{n+1} and r_{n+3}. In this instance, we say that the passage from $\bar{r}$ to $\bar{r}'$ is obtained by a *long braid move*.

Let $w \in W$. Any subset of $\Phi(w)$ of the form $\{\alpha, \beta, \alpha + \beta\}$ is called an *inversion triple* of w. We say that an inversion triple T of w is *contractible* if there is a root sequence for w in which the elements of T appear consecutively (in some order). The number of contractible inversion triples of w is denoted by $N(w)$. If the contractible inversion triples of w are pairwise disjoint, then w is said to be *freely braided*.

Remark 1.1. Suppose that W is of type A_n. Then the freely braided elements $w \in W$ can be characterised by a pattern avoidance condition involving four patterns (this was pointed out by one of the referees, and is also discussed in [**7**, §5.1]). More precisely, a permutation w is freely braided if and only if its 1-line notation avoids 3421, 4231, 4312 and 4321. This fact will not be needed in the sequel.

2. A deletion property of freely braided elements

We introduce a partial order $\leq$ on Φ by writing $\alpha \leq \beta$ if $\beta - \alpha$ is a nonnegative linear combination of simple roots. The following lemma will be used repeatedly in the proof of Proposition 2.2.

Lemma 2.1. *Let $w \in W$ and let $\bar{r}$ be a root sequence for w. Suppose that $\bar{r} = (\ldots, \alpha, \beta, \ldots)$, where α is not orthogonal to β relative to the Coxeter form, and $\alpha \not\leq \beta$. Then $\{\alpha - \beta, \alpha, \beta\}$ is a contractible inversion triple of w.*

Proof. Let $\mathbf{i}ji\mathbf{j}$ be the reduced expression corresponding to $\bar{r}$, parsed in such a way that $\alpha = \phi(\mathbf{j})^{-1}(\alpha_i)$ and $\beta = \phi(\mathbf{j})^{-1}s_i(\alpha_j)$. We first prove that $\{\alpha - \beta, \alpha, \beta\}$ is an inversion triple of w by showing that $\phi(\mathbf{j})$ has a reduced expression starting with j. For this, it suffices by [**10**, Proposition 5.7, Theorem 5.8] to verify that $\phi(\mathbf{j})^{-1}(\alpha_j)$ lies in Φ^-.

Since the root α is not orthogonal to β, we have $m_{ij} = 3$, and hence $s_i(\alpha_j) = \alpha_j + \alpha_i$. We now compute $\phi(\mathbf{j})^{-1}(\alpha_j) = \phi(\mathbf{j})^{-1}(\alpha_j + \alpha_i - \alpha_i) = \phi(\mathbf{j})^{-1}s_i(\alpha_j) - \phi(\mathbf{j})^{-1}(\alpha_i) = \beta - \alpha$, which must lie in Φ^- since $\alpha \not\leq \beta$.

Thus, $\{\alpha - \beta, \alpha, \beta\}$ is an inversion triple of w. Since α and β are adjacent in $\bar{r}$, [**7**, Proposition 3.2.1] now implies that $\{\alpha - \beta, \alpha, \beta\}$ is contractible. $\square$

We define the *height* of any root β to be the sum of the coefficients used to express β as a linear combination of the simple roots.

Proposition 2.2. *Let $w \in W$ be freely braided and suppose that α is the highest root of an inversion triple of w. Then α is the highest root of a contractible inversion triple of w.*

Note. In the type A_n setting, every inversion triple is contractible, but this is not true in general (see [**7**, Example 2.2.3, Proposition 5.1.1]).

Proof. Assume the contrary. Then, by Lemma 2.1, in every root sequence for w, the root α cannot be directly to the left of a root belonging to an inversion triple of w in which α is the highest root. On the other hand, by hypothesis, α is the highest root of some inversion triple of w. Choose a root sequence $(\ldots, \alpha, \beta_1, \ldots, \beta_n, \gamma, \ldots)$ for w with the property that $\{\alpha - \gamma, \alpha, \gamma\}$ is an inversion triple of w, and $n \geq 1$ is as small as possible (over all inversion triples of w having α as highest root).

Now, β_n is not orthogonal to γ by the minimality of n. We claim that $\beta_n \leq \gamma$. Assume otherwise. Then, according to Lemma 2.1, $\{\beta_n - \gamma, \beta_n, \gamma\}$ is a contractible inversion triple of w. By [**7**, Lemma 4.2.2], we can commute $\beta_n - \gamma$ to the right, if necessary, so that it is adjacent to β_n. We cannot have $\beta_n - \gamma = \alpha$, or else, after the commutations, our sequence would take the form $(\ldots, \beta_1, \ldots, \beta_{n-1}, \alpha, \beta_n, \gamma, \ldots)$; a single long braid move would then give us $(\ldots, \beta_1, \ldots, \beta_{n-1}, \gamma, \beta_n, \alpha, \ldots)$, with α lying to the right of both γ and $\alpha - \gamma$, an impossibility [**7**, Remark 2.2.2]. So $\beta_n - \gamma \neq \alpha$. But then our sequence (after the commutations) looks like $(\ldots, \alpha, \beta_1, \ldots, \beta_{n-1}, \beta_n - \gamma, \beta_n, \gamma, \ldots)$, where $\beta_n - \gamma$ may or may not equal one of the β_l. Either way, performing a long braid move gives $(\ldots, \alpha, \beta_1, \ldots, \beta_{n-1}, \gamma, \beta_n, \beta_n - \gamma, \ldots)$, contradicting the minimality of n. The claim is established, i.e., $\beta_n \leq \gamma$.

Let γ^* be a $\leq$-minimal element in $\{\beta_l : \beta_l \leq \gamma\}$. We claim that there is an index m such that β_m is to the left of γ^* and $\beta_m \not\perp \gamma^*$. If such an m did not exist, then we could commute γ^* to the left to obtain the sequence $(\ldots, \alpha, \gamma^*, \ldots, \gamma, \ldots)$. The root γ^* is not orthogonal to α, because if it were, then we could commute it past α and contradict the minimality of n. We also have $\alpha \not\leq \gamma^*$ by the definition of γ^*. Hence, by Lemma 2.1, $\{\alpha - \gamma^*, \alpha, \gamma^*\}$ is a contractible triple, contradicting the fact that α is not the highest root of any contractible triple of w. So there does indeed exist such an m, and we may assume that β_m is directly to the left of γ^* in our chosen root sequence $(\ldots, \alpha, \beta_1, \ldots, \beta_n, \gamma, \ldots)$.

We have $\beta_m \not\leq \gamma^*$ by our choice of γ^*. By Lemma 2.1 again, $\{\beta_m - \gamma^*, \beta_m, \gamma^*\}$ is a contractible triple of w. If necessary, we can commute $\beta_m - \gamma^*$ to the right in our chosen sequence so that it is adjacent to β_m (by [**7**, Lemma 4.2.2]). We can then apply a long braid move so that the three roots of our triple appear in the order $\gamma^*, \beta_m, \beta_m - \gamma^*$. If $\beta_m - \gamma^*$ were to equal α, then this long braid move would place α closer to γ than before, a contradiction. So $\beta_m - \gamma^* \neq \alpha$.

Suppose that $\beta_m - \gamma^*$ was to the left of α before the commutations and long braid move of the previous paragraph. Since we commuted $\beta_m - \gamma^*$ past $\alpha, \beta_1, \ldots, \beta_{m-1}$ in order to put it next to β_m, it is orthogonal to all of these vectors. We claim that the root γ^* is also orthogonal to $\alpha, \beta_1, \ldots, \beta_{m-1}$. To see this, assume otherwise. Then, after the long braid move of the previous paragraph, we can commute γ^* to the left until it is adjacent to a root δ among $\alpha, \beta_1, \ldots, \beta_{m-1}$ with which it is not orthogonal. Since γ^* cannot be $\geq \delta$, Lemma 2.1 implies that γ^* and δ belong to the same contractible triple of w. This is a contradiction, since γ^* cannot belong to any contractible triple of w other than $\{\beta_m - \gamma^*, \beta_m, \gamma^*\}$, owing to the fact that w is freely braided. The claim is established. It follows that each root in $\{\beta_m - \gamma^*, \beta_m, \gamma^*\}$ is orthogonal to $\alpha, \beta_1, \ldots, \beta_{m-1}$, and so can be commuted to the left past all of them, contradicting the minimality of n.

Suppose finally that $\beta_m - \gamma^*$ is one of the β_l. Even here, after applying the long braid move mentioned two paragraphs above, we can commute γ^* to the left past α (by the argument of the previous paragraph), contradicting the minimality of n. This last contradiction exhausts all possibilities for $\beta_m - \gamma^*$. $\square$

Let $\mathbf{i}$ be a word in I^* and suppose that $\mathbf{i}$ can be written as $\mathbf{u}_0\mathbf{b}_1\mathbf{u}_1\mathbf{b}_2\mathbf{u}_2\cdots\mathbf{b}_p\mathbf{u}_p$, where each $\mathbf{b}_l$ is of the form iji for some $i, j \in I$ with $m_{ij} = 3$. Then we call $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ a *braid sequence* for $\mathbf{i}$. If $\mathbf{i} \in I^*$ is reduced and $w = \phi(\mathbf{i})$ is freely braided, then we say that $\mathbf{i}$ is *contracted* provided there exists a braid sequence for $\mathbf{i}$ with $p = N(w)$ terms.

Proposition 2.3 [**8**, Proposition 3.1.3]**.** *Let $w \in W$ be freely braided.*

(i) *There exists a contracted reduced expression for w.*

(ii) *Any contracted reduced expression for w has a unique braid sequence with $N(w)$*

terms. □

The proof of the next proposition adapts an argument of Fan [**6**, Theorem 1]. We continue to assume that W is simply laced.

Proposition 2.4. *Assume that W is finite. Let w be a freely braided element of W, and let $\mathbf{i} = i_1 \cdots i_n$ be a contracted reduced expression for w with braid sequence $\mathbf{b}_1, \ldots, \mathbf{b}_{N(w)}$. Then the expression obtained from $\mathbf{i}$ by deleting any letter, except a middle letter from one of the braids $\mathbf{b}_l$, is reduced.*

Proof. Suppose that the deletion of a letter i_m from $\mathbf{i}$ results in a word $\mathbf{i}'$ that is not reduced. We will show that i_m must be a middle letter of one of the braids $\mathbf{b}_l$. Denote by y the element of W represented by $i_{m+1} \cdots i_n$, and note that $y^{-1}(\alpha_{i_m}) \in \Phi(w)$. Since $\mathbf{i}$ is contracted, we need only show that $y^{-1}(\alpha_{i_m})$ is the highest root of a contractible triple of w. By Proposition 2.2, it suffices to show that $y^{-1}(\alpha_{i_m})$ is the highest root of a (not necessarily contractible) inversion triple of w.

There is an index p such that the expression $i_{p+1} \cdots i_{m-1} i_{m+1} \cdots i_n$ is reduced and $i_p \cdots i_{m-1} i_{m+1} \cdots i_n$ is not. Let x be the element of W represented by the word $i_{p+1} \cdots i_{m-1}$, and let α be the root $x^{-1}(\alpha_{i_p})$, which is positive since $i_p \cdots i_{m-1}$ is reduced.

Let $\gamma_1 = y^{-1}(\alpha + \alpha_{i_m})$, $\gamma_2 = -y^{-1}(\alpha)$ and $\gamma_3 = y^{-1}(\alpha_{i_m})$. Clearly, $\gamma_1 + \gamma_2 = \gamma_3$. We claim that this is an inversion triple of the element w' represented by $i_p \cdots i_m \cdots i_n$. Since $i_{p+1} \cdots i_{m-1} i_{m+1} \cdots i_n$ is reduced and $i_p \cdots i_{m-1} i_{m+1} \cdots i_n$ is not, $y^{-1} x^{-1}(\alpha_{i_p})$ is negative. This root equals $y^{-1}(\alpha)$, hence γ_2 is positive.

Next, we show that γ_1 is positive. Let $c = B(\alpha, \alpha_{i_m})$. Note that $\alpha \neq \alpha_{i_m}$ (to see this, recall from above that $y^{-1}(\alpha_{i_m})$ is positive whereas $y^{-1}(\alpha)$ is negative). Therefore, since W is finite, c can only equal $-1/2$, 0, or $1/2$. If c were to equal 0, then $y^{-1} s_{i_m}(\alpha)$ would equal $y^{-1}(\alpha)$, a negative root; this would contradict the fact that $y^{-1} s_{i_m}(\alpha)$ belongs to the inversion set of w. If c were to equal $1/2$, then the positive root $y^{-1} s_{i_m}(\alpha)$ would equal $y^{-1}(\alpha - \alpha_{i_m}) = y^{-1}(\alpha) - y^{-1}(\alpha_{i_m})$, which is negative, another contradiction. Thus, $c = -1/2$, and this implies that the positive root $y^{-1}s_{i_m}(\alpha)$ equals $y^{-1}(\alpha + \alpha_{i_m}) = \gamma_1$.

We verify that the roots γ_1 and γ_2 are sent negative by w'. This follows from the calculations $w'(\gamma_1) = s_{i_p}xs_{i_m}yy^{-1}(\alpha + \alpha_{i_m}) = s_{i_p}x(\alpha) = -\alpha_{i_p}$ and $w'(\gamma_2) = -s_{i_p}xs_{i_m}yy^{-1}(\alpha) = -s_{i_p}x(\alpha + \alpha_{i_m}) = -s_{i_p}(\alpha_{i_p} + x(\alpha_{i_m})) = \alpha_{i_p} - s_{i_p}x(\alpha_{i_m})$. Regarding the latter, since α_{i_p} is simple and $s_{i_p}x(\alpha_{i_m})$ is positive (because the word $i_p \cdots i_m$ is reduced), we have that $w'(\gamma_2)$ is negative.

Thus, $\{\gamma_1, \gamma_2, \gamma_3\}$ is an inversion triple of w'. Since every inversion triple of w' is also an inversion triple of w, this proves that $y^{-1}(\alpha_{i_m})$ is the highest root of some inversion triple of w, as desired. $\square$

Remark 2.5. Let $w \in W$ be freely braided. By [**8**, Proposition 3.1.3(ii)], any reduced expression for w can be transformed into a contracted expression by performing a sequence of short braid moves. Thus, in the case where W is finite, Proposition 2.4 can actually be used to decide whether the deletion of a letter from an arbitrary reduced expression for w results in another reduced expression.

3. Application to Schubert varieties

Throughout this section, we assume that our simply laced group W is the Weyl group of a semisimple simply-connected complex algebraic group G with fixed maximal torus T and Borel subgroup $B \supset T$. For each $w \in W$, let X_w be the associated Schubert variety, i.e., the closure of the cell BwB/B in the generalized flag variety G/B. Define, for each $w \in W$, the polynomial $P_w(t) = \sum_{v \preceq w} t^{\ell(v)}$, where $\preceq$ denotes the Bruhat–Chevalley partial order on W. Then $P_w(t^2)$ is the Poincaré polynomial for the cohomology ring of X_w.

Carrell–Peterson have shown that a Schubert variety X_w is rationally smooth if and only if the polynomial $P_w(t)$ is symmetric, meaning $P_w(t) = t^{\ell(w)}P_w(1/t)$ [**4**]. Note that in the simply laced setting, rational smoothness and smoothness are equivalent for Schubert varieties (this is an unpublished result of Peterson; see [**5**]).

Let $w \in W$. The set of letters from I appearing in some (any) reduced expression for w will be called the *support* of w, and will be denoted by $\mathrm{supp}(w)$. If w is freely braided, then it has a contracted reduced expression by Proposition 2.3. Consideration of this reduced expression reveals that $\ell(w) - N(w) \geq \# \mathrm{supp}(w)$.

Definition 3.1. If w is freely braided and $\# \mathrm{supp}(w) = \ell(w) - N(w)$, then w is said to be *content maximal*.

Let $v, w \in W$ and let $\mathbf{i} = i_1 \cdots i_n$ be a reduced expression for w. Recall that $v \preceq w$ if and only if v is represented by a (possibly empty) word of the form $i_{\mu_1} \cdots i_{\mu_m}$, where $1 \leq \mu_1 < \cdots < \mu_m \leq n$ [**10**, Theorem 5.10].

Theorem 3.2. *Let $w \in W$ be freely braided. Then the Schubert variety X_w is smooth if and only if w is content maximal.*

Proof. By Proposition 2.4 and [**6**, Lemma 2], there are exactly $\ell(w) - N(w)$ elements of W having length $\ell(w) - 1$ that are less than w in the Bruhat–Chevalley order. If X_w is (rationally) smooth, then $P_w(t)$ is symmetric, and it follows that there are exactly $\ell(w) - N(w)$ elements of length 1 that are less than w, as required.

Now assume that the support of w has $\ell(w) - N(w)$ elements.

Let $\mathbf{i} = \mathbf{u}_0 \mathbf{b}_1 \mathbf{u}_1 \mathbf{b}_2 \mathbf{u}_2 \cdots \mathbf{b}_{N(w)} \mathbf{u}_{N(w)}$ be a contracted reduced expression for w. By content maximality, the only letters appearing more than once in $\mathbf{i}$ are the outer factors of the braids $\mathbf{b}_l$. We will show that the polynomial $P_w(t)$ is symmetric, and the theorem will follow from the results of Carrell and Peterson cited above.

We argue by induction on $d = \ell(w)$. For $d \leq 3$, the polynomial $P_w(t)$ is symmetric by direct verification, so we assume $d > 3$. There are two cases to consider.

Case 1: $\mathbf{u}_{N(w)}$ is a nonempty word, ending in the letter i.

Let $\mathbf{i}'$ be the reduced expression obtained from $\mathbf{i}$ by deleting i, and let v be the group element represented by $\mathbf{i}'$. Now, i appears exactly once in $\mathbf{i}$, because it does not appear in a braid. Therefore, by [**8**, Lemma 3.2.2], v is freely braided and

$N(v) = N(w)$. We thus also have $\# \operatorname{supp}(v) = \ell(v) - N(v)$, so that v satisfies the inductive hypothesis.

Since i does not appear in $\mathbf{i}'$, the set $\{x : x \preceq w\}$ equals the disjoint union of the sets $L_1 = \{x : x \preceq v\}$ and $L_2 = \{xs_i : x \preceq L_1\}$. Furthermore, for $x \in L_1$, we have $\ell(xs_i) = \ell(x) + 1$. It follows that $P_w(t) = (1+t)P_v(t)$. Since $\deg P_v(t) = \deg P_w(t) - 1$ and $P_v(t)$ is symmetric by induction, $P_w(t)$ is symmetric.

Case 2: $\mathbf{u}_{N(w)}$ is empty.

Since $d > 3$, we have $N(w) \geq 1$. Write $\mathbf{b}_{N(w)} = iji$ and let $v = ws_is_js_i$. Since i and j do not lie in the support of v, any element $x \preceq w$ is uniquely of the form $x = ab$ where $a \preceq v$, $b \preceq s_is_js_i$ and $\ell(ab) = \ell(a) + \ell(b)$.

It follows that $P_w(t) = (1+t)(1+t+t^2)P_v(t)$. By [**8**, Lemma 3.2.2] again (applied three times successively to w), v is freely braided and $N(v) = N(w) - 1$. Since $\ell(v) = \ell(w) - 3$ and $\# \operatorname{supp}(v) = \# \operatorname{supp}(w) - 2$, the inductive hypothesis is satisfied by v, so that $P_v(t)$ is symmetric. Since $\deg P_v(t) = \deg P_w(t) - 3$, the induction is complete. $\square$

Remark 3.3. Another possible approach to proving Theorem 3.2 would be to use the criterion for smoothness recently found by Billey–Postnikov [**3**].

An element $w \in W$ is said to be *fully commutative* if every reduced expression for w can be transformed into any other by performing a sequence of short braid moves [**14**]. This is equivalent to the requirement that $N(w) = 0$ [**8**, Proposition 1.2.2]. Thus, every fully commutative element of W is freely braided. Note that the fully commutative elements coincide with the "short-braid avoiding" elements of [**6**] in the case where W is simply laced (the only case considered in this paper).

We have the following immediate corollary of Theorem 3.2, which was originally proved by Fan for short-braid avoiding elements of an arbitrary Weyl group [**6**, Proposition 3].

Corollary 3.4. *Let $w \in W$ be fully commutative. Then the Schubert variety X_w*

is smooth if and only if w equals a product of $\ell(w)$ distinct generators. $\square$

4. Enumeration of content maximal elements

Motivated by Theorem 3.2, we derive generating functions for the number of content maximal elements in types A_n, D_n and E_n. In the E-series, we allow n to be any positive integer ≥ 6, since all groups in this series have finitely many freely braided elements [**8**, Theorem 3.3.3] and our methods apply equally well to them.

It turns out that all three generating functions are based on a single recurrence relation; we explain by using the following set-up. Consider a nested sequence of simply laced Coxeter groups $W_1 \subset W_2 \subset \cdots$ with generating sets $S_1 \subset S_2 \subset \cdots$ and alphabets $I_1 \subset I_2 \subset \cdots$. Thus, for each $n \geq 1$, we have $S_n = \{s_i : i \in I_n\}$. Assume that for any positive integers $m < n$ and any $i, j \in I_m$, the order of $s_i s_j$ is the same in W_m as it is in W_n. Assume further the existence of a sequence $(i_n)_{n \geq 1}$ such that each i_n lies in I_n, and whenever $n > 1$, we have $m_{i_n j} = 2$ for all $j \in I_{n-1}$ except for $j = i_{n-1}$.

Some ad hoc terminology will be useful. Given a content maximal element w and a letter i, we say that i is *not braided* in w if i appears at most once in any reduced expression for w. If i appears more than once in some reduced expression for w, then by content maximality it appears exactly twice and there is precisely one letter j between the two occurrences of i such that $m_{ij} = 3$. In this case, we say that i is *braided* in w *with* j. It is clear that if i is braided (in w) with j, then j is braided with i. Note that if i is braided in w, then it is braided with precisely one other letter.

Let $w \in W$. In the proof of the following lemma, we will use a result of Matsumoto [**13**] and Tits [**15**], which states that any reduced expression for w can be transformed into any other by applying a sequence of long and short braid moves.

Denote by $f(n)$ the number of content maximal elements in W_n.

Lemma 4.1. *Let $w \in W_n$ be content maximal. Then w satisfies one of the follow-*

ing seven mutually exclusive conditions:

(i) $i_n \notin \mathrm{supp}(w)$;

(ii) $i_n \in \mathrm{supp}(w)$ *and* i_n *is not braided in* w *and if* $n > 1$*, then* $i_{n-1} \notin \mathrm{supp}(w)$;

(iii) $i_n \in \mathrm{supp}(w)$ *and* i_n *is not braided in* w *and* $i_{n-1} \in \mathrm{supp}(w)$ *and* $\ell(ws_{i_n}) < \ell(w)$;

(iv) $i_n \in \mathrm{supp}(w)$ *and* i_n *is not braided in* w *and* $i_{n-1} \in \mathrm{supp}(w)$ *and* $\ell(s_{i_n}w) < \ell(w)$;

(v) $i_n \in \mathrm{supp}(w)$ *and* i_n *is braided in* w *and if* $n > 2$*, then* $i_{n-2} \notin \mathrm{supp}(w)$;

(vi) $i_n \in \mathrm{supp}(w)$ *and* i_n *is braided in* w *and* $i_{n-2} \in \mathrm{supp}(w)$ *and* $\ell(ws_{i_{n-1}}) < \ell(w)$;

(vii) $i_n \in \mathrm{supp}(w)$ *and* i_n *is braided in* w *and* $i_{n-2} \in \mathrm{supp}(w)$ *and* $\ell(s_{i_{n-1}}w) < \ell(w)$.

For $n > 3$*, the number of content maximal elements of* W_n *satisfying the respective conditions* (i), (ii), ..., (vii) *is* $f(n-1)$, $f(n-2)$, $f(n-1) - f(n-2)$, $f(n-1) - f(n-2)$, $f(n-3)$, $f(n-2) - f(n-3)$, *and* $f(n-2) - f(n-3)$. *We therefore have the recurrence* $f(n) = 3f(n-1) + f(n-2) - f(n-3)$.

Note. In conditions (iii)–(v), we are assuming $n > 1$. In conditions (vi) and (vii), we are assuming $n > 2$.

Proof. Suppose that $i_n \in \mathrm{supp}(w)$ and i_n is not braided in w and $i_{n-1} \in \mathrm{supp}(w)$. We claim that either $\ell(ws_{i_n}) < \ell(w)$ or $\ell(s_{i_n}w) < \ell(w)$. Fix a contracted reduced expression for w. By our assumptions, i_n appears exactly once and i_{n-1} appears at least once and at most twice in this expression. If i_{n-1} appears only once, then, since $m_{i_n j} = 2$ for all $j \in I_{n-1}$ except for $j = i_{n-1}$, we can commute i_n to one of the ends of our contracted expression. If instead i_{n-1} appears twice, then it is braided, necessarily with i_{n-2}. Once again, we can commute i_n to an end of our contracted expression. Thus, either (iii) or (iv) holds. Note that these conditions cannot hold simultaneously, by our assumption that i_n is not braided in w together with the result of Matsumoto and Tits cited above.

Similar reasoning shows that if $i_n \in \mathrm{supp}(w)$ and i_n is braided in w and $i_{n-2} \in \mathrm{supp}(w)$, then exactly one of the conditions (vi) and (vii) holds. Thus, our seven conditions are mutually exclusive and collectively exhaustive.

Regarding the cardinality assertions, let C_n denote the set of content maximal elements in W_n and let $C_n(\mathrm{i})$, $C_n(\mathrm{ii})$, ... , $C_n(\mathrm{vii})$ denote the respective subsets of elements satisfying conditions (i), (ii), ... , (vii). We have $C_n(\mathrm{i}) = C_{n-1}$, so that $\#C_n(\mathrm{i}) = f(n-1)$. The mapping $C_n(\mathrm{ii}) \longrightarrow C_{n-2}$ given by $w \mapsto ws_{i_n}$ is a well-defined bijection, hence $\#C_n(\mathrm{ii}) = f(n-2)$. Similarly, there are bijections $C_n(\mathrm{iii}) \longrightarrow C_{n-1} \setminus C_{n-2}$ and $C_n(\mathrm{iv}) \longrightarrow C_{n-1} \setminus C_{n-2}$, given by $w \mapsto ws_{i_n}$ and $w \mapsto s_{i_n}w$, respectively. It follows that $\#C_n(\mathrm{iii}) = \#C_n(\mathrm{iv}) = f(n-1) - f(n-2)$. The remaining bijections are induced by left or right multiplication by $s_{i_n}s_{i_{n-1}}s_{i_n}$. □

Through direct computation, using either a variation of the proof of the above lemma or distinguished coset representatives for the subgroup W_{n-1} of W_n, one finds that the respective numbers of content maximal elements in Coxeter groups of type A_1, A_2, A_3, D_4, D_5 and E_6 are 2, 6, 19, 62, 201 and 652. This information, together with Lemma 4.1 itself, enables one to derive the ordinary generating functions for the numbers $f(n)$ when W_n is of type A_n or D_n or E_n, as follows.

Theorem 4.2. *The ordinary generating function for the number of content maximal elements is*

$$\begin{cases} \dfrac{2x - x^3}{1 - 3x - x^2 + x^3} = 2x + 6x^2 + 19x^3 + 61x^4 + 196x^5 + \cdots & \textit{in type } A_n, \\ \dfrac{62x^4 + 15x^5 - 19x^6}{1 - 3x - x^2 + x^3} = 62x^4 + 201x^5 + 646x^6 + 2077x^7 + \cdots & \textit{in type } D_n, \\ \dfrac{652x^6 + 140x^7 - 201x^8}{1 - 3x - x^2 + x^3} = 652x^6 + 2096x^7 + 6739x^8 + \cdots & \textit{in type } E_n. \end{cases}$$

□

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