A STRAIGHTENING FORMULA FOR QUANTIZED CODETERMINANTS

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0. Introduction

J.A. Green [**6**, §4] introduces *codeterminants*, which are products of pairs of natural basis elements of the Schur algebra $S(n,r)$. These are shown to be parametrised by pairs of row-semistandard tableaux of shape λ, where $\lambda \in \Lambda(n,r)$. If both the tableaux are standard, and the composition λ is dominant, then the codeterminant is said to be standard. The set of standard codeterminants forms a free $\mathbf{Z}$-basis for the integral form of the classical Schur algebra. An algorithm for expressing an arbitrary codeterminant as a sum of standard ones has been described in [**9**].

It is natural to ask whether these results quantize. In this paper, we prove that they do. The Schur algebra has a quantum analogue, the so-called *q-Schur algebra*, $S_q(n,r)$, which is defined over the ring $\mathbf{Z}[q,q^{-1}]$. We will also be working with the v-Schur algebra, which is $\mathcal{A} \otimes S_q(n,r)$, where $\mathcal{A} = \mathbf{Z}[v,v^{-1}]$ and $v^2 = q$. An algorithm to compute products in these algebras was established in [**7**]. We define codeterminants for these algebras in an analogous way to Green [**6**], and after a careful analysis of certain products in $S_v(n,r)$, we state and prove a straightening formula for them, and exhibit a free basis of codeterminants for $S_q(n,r)$.

The rest of this paper falls into three sections. §1 consists of definitions and a summary of some of the important results in [**7**]. In §2, we show that any codeterminant can be expressed as a $\mathbf{Z}[q,q^{-1}]$-linear sum of codeterminants of dominant shape. In §3, we prove a quantum analogue of the straightening formula for codeterminants. This is a procedure for expressing any codeterminant of dominant shape as a $\mathbf{Z}[q,q^{-1}]$-linear sum of standard codeterminants. This was proved by Woodcock [**9**] in the classical case. We use some of the techniques in that paper to prove the quantum analogue, but some steps are considerably more delicate and require new methods.

1. Preliminaries

1.1 The Classical Schur Algebra

The classical Schur algebra, $S(n,r)$ has a basis consisting of certain elements $\xi_{\mathbf{i},\mathbf{j}}$ which are defined (as in Green [**5**, §2.3]) as follows.

Let $I(n,r)$ be the set of all ordered r-tuples of elements from the set $\mathbf{n} := \{1,\ldots,n\}$. (Elements of $I(n,r)$ will be denoted by bold letters or by using the symbol ℓ, to avoid confusion with integers. The elements of $\mathbf{i} \in I$ will be denoted by i_1, i_2 etc.) The symmetric group $\mathcal{S}_r$ acts on the set $I = I(n,r)$ on the right by place permutation via

$$(i_1,\ldots,i_r).\pi := (i_{(1)\pi},\ldots,i_{(r)\pi}).$$

It also acts on the set $I \times I$ as $(\mathbf{i},\mathbf{j})\pi = (\mathbf{i}\pi,\mathbf{j}\pi)$. We write $\mathbf{i} \sim \mathbf{j}$ if $\mathbf{i}$ and $\mathbf{j}$ are in the same $\mathcal{S}_r$-orbit of I, and $(\mathbf{i},\mathbf{j}) \sim (\mathbf{i}',\mathbf{j}')$ if $(\mathbf{i},\mathbf{j})$ and $(\mathbf{i}',\mathbf{j}')$ are in the same $\mathcal{S}_r$-orbit of $I \times I$. The subscripts $\mathbf{i}$ and $\mathbf{j}$ of the element $\xi_{\mathbf{i},\mathbf{j}}$ lie in $I(n,r)$, and we identify $\xi_{\mathbf{i},\mathbf{j}}$ and $\xi_{\mathbf{i}',\mathbf{j}'}$ if and only if $(\mathbf{i},\mathbf{j}) \sim (\mathbf{i}',\mathbf{j}')$. The set of all $\xi_{\mathbf{i},\mathbf{j}}$, as $(\mathbf{i},\mathbf{j})$ ranges over a transversal Ω of all $\mathcal{S}_r$-orbits of $I \times I$ can be shown (see e.g. [**5**, §2.3]) to be a basis for $S(n,r)$. We will usually write $\xi_{\mathbf{l}}$ as shorthand for $\xi_{\mathbf{l},\mathbf{l}}$.

Schur's product rule for the basis $\{\xi_{\mathbf{i},\mathbf{j}}\}$ is given by

$$\xi_{\mathbf{i},\mathbf{j}}\xi_{\mathbf{k},\mathbf{l}} = \sum_{\mathbf{p},\mathbf{q}\in\Omega} Z(\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l},\mathbf{p},\mathbf{q})\xi_{\mathbf{p},\mathbf{q}},$$

where $Z(\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l},\mathbf{p},\mathbf{q}) := |\{\mathbf{s} \in I : (\mathbf{i},\mathbf{j}) \sim (\mathbf{p},\mathbf{s}) \text{ and } (\mathbf{k},\mathbf{l}) \sim (\mathbf{s},\mathbf{q})\}|$.

The basis for $S(n,r)$ described above can be thought of in a different way, this time indexed by the set Θ_r of $n\times n$ matrices with nonnegative integer entries summing to r. A typical basis element will be denoted by e_A, where A is the matrix in question.

Let $\mathbf{i}=(i_1,\ldots,i_r)$ and $\mathbf{j}=(j_1,\ldots,j_r)$ be elements of I. The correspondence between the two bases is given by

$$\xi_{\mathbf{i},\mathbf{j}}=e_A,$$

where the (x,y)-entry of the matrix A is given by the number of pairs (i_s,j_s) such that $i_s=x$ and $j_s=y$.

1.2 The parameters q and v

Following Du [**4**], we define $\mathcal{A}:=\mathbf{Z}[v,v^{-1}]$, and $q=v^2$. It will often be the case in the proofs that follow that it is easier initially to work over $\mathcal{A}$ and deduce the corresponding results involving $\mathbf{Z}[q,q^{-1}]$ from them. The reason for this is that the basis $\{e_A\}$ for $S_q(n,r)$ is free over the ring $\mathbf{Z}[v,v^{-1}]$, and hence over $\mathbf{Z}[q,q^{-1}]$. The motivation behind casting the results into the form involving q is that when the q-Schur algebra is studied in its own right, it is often the case that we do not need to introduce v, which is a square root of q. We will use the notation $S_v(n,r)$ when we wish to emphasise the fact that we are working over $\mathcal{A}$.

Sometimes we will write v^* to denote some integer power of v, but usually no attempt will be made to keep track of the exact value of this quantity, because it isn't important for the purposes of the proof. The value of v^* may change from line to line during the proof.

1.3 The q-Schur algebra, $S_q(n,r)$

We now define (following [**4**] or [**1**, Proposition 1.2]) the q-Schur algebra, $S_q(n,r)$, to be the $\mathbf{Q}(v)$-vector space with basis $\{e_A : A\in\Theta_r\}$ with associative multiplication given by

$$e_A e_{A'}=\sum_{A''\in\Theta_r} g_{A,A',A''}e_{A''},$$

where the structure constants $g_{A,A',A''}$ are certain polynomials in $\mathbf{Z}[q]$ arising from the geometry of flags.

Du [**4**] remarks that this algebra is canonically isomorphic to the q-Schur algebra defined by Dipper and James, by exhibiting the following correspondence between basis elements e_A and basis elements $\phi^d_{\lambda\mu}$ as defined by Dipper and James in [**3**]. Here, the elements λ and

μ lie in $\Lambda(n,r)$, which is the set of compositions of r into n parts, and $d \in \mathcal{D}_{\lambda\mu}$, which is the set of distinguished W_λ–W_μ double coset representatives for the Young subgroups W_λ and W_μ of $\mathcal{S}_r$, the symmetric group on r letters. Suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$. For each $\alpha \in \mathbf{n}$, we define I_α to be the subset of $\mathbf{n}$ given by

$$I_\alpha := \{\lambda_1 + \cdots + \lambda_{\alpha-1} + 1, \lambda_1 + \cdots + \lambda_{\alpha-1} + 2, \ldots, \lambda_1 + \cdots + \lambda_\alpha\}.$$

Similarly, we define J_α to be the analogous subset of $\mathbf{n}$ corresponding to μ and α. Given a Dipper-James basis element, $\phi^d_{\lambda\mu}$, we define a corresponding matrix A via

$$A_{\alpha\beta} := |d(J_\beta) \cap I_\alpha|.$$

This procedure sets up the required isomorphism by sending $\phi^d_{\lambda\mu}$ to e_A. It should be noted that λ corresponds to the sums of the rows of A, and μ to the sums of the columns of A.

In this paper, we will prefer to think of these basis elements e_A as elements $\xi_{\mathbf{i},\mathbf{j}}$ (as in the classical case), where the e_A and $\xi_{\mathbf{i},\mathbf{j}}$ are identified as in §1.1.

1.4 Coxeter Groups and Hecke Algebras

We consider $W = \mathcal{S}_r$, the symmetric group on r letters, as a Coxeter group with generators $S = \{s_1, \ldots, s_{r-1}\}$ (where s_i is the transposition $(i, i+1)$) and defining relations

$$\begin{aligned} s_i^2 &= 1; \\ s_i s_j &= s_j s_i \quad \text{if } |i-j| > 1; \\ s_i s_j s_i &= s_j s_i s_j \quad \text{if } |i-j| = 1. \end{aligned}$$

This makes (W, S) into a Coxeter system, and W into a Coxeter group.

Define the *length* of $w \in W$, denoted by $\ell(w)$, to be the smallest r for which an expression $w = s_{i_1} \cdots s_{i_r}$ exists. Such a minimal expression for w is called a *reduced* expression. It is well-known that the length of w is also given by the cardinality of the set

$$\{\langle i,j\rangle : i < j, (i)w > (j)w\}.$$

Thus we have the following identities:

$$\ell(s_i w) = \begin{cases} \ell(w) + 1 & \text{if } iw < (i+1)w; \\ \ell(w) - 1 & \text{if } iw > (i+1)w; \end{cases}$$
$$\ell(w s_i) = \begin{cases} \ell(w) + 1 & \text{if } iw^{-1} < (i+1)w^{-1}; \\ \ell(w) - 1 & \text{if } iw^{-1} > (i+1)w^{-1}. \end{cases}$$

We now introduce the *Hecke algebra* $\mathcal{H}(W)$ corresponding to the Coxeter group W. This has basis elements T_w parametrised by $w \in W$, and the following multiplication rules:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w); \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

The existence of this algebra structure is proved in [**8**, §7].

It follows from the relations that if $w = s_{i_1} \cdots s_{i_r}$ is reduced, then $T_w = T_{s_1} \cdots T_{s_r}$. It also follows that if $\ell(uw) = \ell(u) + \ell(w)$ then $T_{uw} = T_u T_w$.

Notice that when q is replaced by 1, the algebra $\mathcal{H} = \mathcal{H}(W)$ is canonically isomorphic to the group algebra of W.

The following property of Hecke algebras will be used in this paper.

Lemma 1.1 Let $d \in W$. Then

$$T_d . \left(\sum_{w \in W} T_w \right) = q^{\ell(d)} . \left(\sum_{w \in W} T_w \right).$$

Proof This is well-known (see [**2**, §3.2]) and follows by induction on $\ell(d)$. ∎

Let W' be a subgroup of a Coxeter group W. Define

$$x_{W'} := \sum_{w \in W'} T_w.$$

1.5 Tensor Space

Let V be the $\mathbf{Q}(v)$-vector space with basis $\{e_1, \ldots, e_n\}$. Following [**4**], we make $T^r(V) = V^{\otimes r}$ into a right $\mathcal{H}$-module by defining $e_{i_1} \otimes \cdots \otimes e_{i_r} . T_{p,p+1}$ as

$$\begin{cases} ve_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r} & \text{if } i_p < i_{p+1}; \\ v^2 e_{i_1} \otimes \cdots \otimes e_{i_r} & \text{if } i_p = i_{p+1}; \\ ve_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r} + (v^2 - 1)e_{i_1} \otimes \cdots \otimes e_{i_r} & \text{if } i_p > i_{p+1}. \end{cases}$$

We will also write $u_{\mathbf{i}}$ for $e_{i_1} \otimes \cdots \otimes e_{i_r}$, where $\mathbf{i}$ is the element of $I(n,r)$ given by $(i_1, \ldots, i_r)$.

Define the function $m : I(n,r) \to \mathbf{Z}$ by

$$m(\mathbf{i}) := |\{\langle a, b \rangle : 1 \leq a < b \leq r : i_a > i_b\}|.$$

We will write $[u_{\mathbf{i}}]$ for $v^{-m(\mathbf{i})} u_{\mathbf{i}}$.

The action of $\mathcal{H}$ can now be re-expressed as follows. Let $s = (p, p+1)$. Then

$$[u_{\mathbf{i}}].T_s = \begin{cases} q[u_{\mathbf{i}.s}] & \text{if } i_p \leq i_{p+1}; \\ [u_{\mathbf{i}.s}] + (q-1)[u_{\mathbf{i}}] & \text{if } i_p > i_{p+1}. \end{cases}$$

The action of T_s^{-1} is also easy to describe:

$$[u_{\mathbf{i}}].T_s^{-1} = \begin{cases} q^{-1}[u_{\mathbf{i}.s}] & \text{if } i_p \geq i_{p+1}; \\ [u_{\mathbf{i}.s}] + (q^{-1}-1)[u_{\mathbf{i}}] & \text{if } i_p < i_{p+1}. \end{cases}$$

1.6 Tensor Matrix Space

Consider the $\mathbf{Q}(v)$-vector space with basis $\{u_{\mathbf{i},\mathbf{j}} : i, j \in I(n,r)\}$. This has an associative multiplication defined on it by $u_{\mathbf{i},\mathbf{j}} u_{\mathbf{k},\mathbf{l}} := \delta_{\mathbf{j},\mathbf{k}} u_{\mathbf{i},\mathbf{l}}$. We call the resulting algebra *tensor matrix space*, and denote it by $T^r(M_n)$, because it is canonically isomorphic to the r-th tensor power of the matrix ring $M_n(\mathbf{Q}(v))$ via the map σ defined by

$$\sigma(u_{\mathbf{i},\mathbf{j}}) = e_{i_1 j_1} \otimes \cdots \otimes e_{i_r j_r}.$$

The algebra $T^r(M_n)$ has another basis, which we shall call the *quantized basis*, consisting of the set $\{[u_{\mathbf{i},\mathbf{j}}] : \mathbf{i}, \mathbf{j} \in I(n,r)\}$. The element $[u_{\mathbf{i},\mathbf{j}}]$ is defined to be $v^{-m(\mathbf{i})-m(\mathbf{j})} u_{\mathbf{i},\mathbf{j}}$ where the function m is as in 1.5.

It was shown in [**7**, Lemma 3.1] that $T^r(M_n)$ admits a right $\mathcal{H}$-module structure which we now describe. Let $T_s = T_{(p,p+1)}$. Then:

$$[u_{\mathbf{i},\mathbf{j}}].T_s = \begin{cases} q[u_{\mathbf{i}.s,\mathbf{j}.s}] & \text{if } i_p \leq i_{p+1} \text{ and } j_p \leq j_{p+1}; \qquad (1) \\ [u_{\mathbf{i}.s,\mathbf{j}.s}] + (q-1)[u_{\mathbf{i},\mathbf{j}}] & \text{if } i_p > i_{p+1} \text{ and } j_p > j_{p+1}; \qquad (2) \\ [u_{\mathbf{i}.s,\mathbf{j}.s}] + (q-1)[u_{\mathbf{i},\mathbf{j}.s}] & \text{if } i_p > i_{p+1} \text{ and } j_p \leq j_{p+1}; \qquad (3) \\ q[u_{\mathbf{i}.s,\mathbf{j}.s}] + (q-1)[u_{\mathbf{i},\mathbf{j}}] - (q-1)[u_{\mathbf{i},\mathbf{j}.s}] & \text{if } i_p \leq i_{p+1} \text{ and } j_p > j_{p+1}. \qquad (4) \end{cases}$$

From this, the action of $T_{(p,p+1)}^{-1}$ immediately follows. We find that $[u_{\mathbf{i},\mathbf{j}}].T_s^{-1}$ is given by

$$\begin{cases} q^{-1}[u_{\mathbf{i}.s,\mathbf{j}.s}] & \text{if } i_p \geq i_{p+1} \text{ and } j_p \geq j_{p+1}; \qquad (5) \\ [u_{\mathbf{i}.s,\mathbf{j}.s}] + (q^{-1}-1)[u_{\mathbf{i},\mathbf{j}}] & \text{if } i_p < i_{p+1} \text{ and } j_p < j_{p+1}; \qquad (6) \\ [u_{\mathbf{i}.s,\mathbf{j}.s}] + (q^{-1}-1)[u_{\mathbf{i},\mathbf{j}.s}] & \text{if } i_p < i_{p+1} \text{ and } j_p \geq j_{p+1}; \qquad (7) \\ q^{-1}[u_{\mathbf{i}.s,\mathbf{j}.s}] + (q^{-1}-1)[u_{\mathbf{i},\mathbf{j}}] - (q^{-1}-1)[u_{\mathbf{i},\mathbf{j}.s}] & \text{if } i_p \geq i_{p+1} \text{ and } j_p < j_{p+1}. \qquad (8) \end{cases}$$

Note

Notice that this action specialises (when q and v are replaced by 1) to permutation, and that any basis element occurring with nonzero coefficient in $u_{\mathbf{a},\mathbf{b}}.T_w$ is of form $u_{\mathbf{c},\mathbf{d}}$ where $\mathbf{a} \sim \mathbf{c}$ and $\mathbf{b} \sim \mathbf{d}$.

1.7 Parabolic subgroups

Consider a Coxeter system (W, S). For any subset $I \subset S$, we can define a subgroup W_I of W, generated by the set I. This is called a *parabolic subgroup* of W. Parabolic subgroups have various important properties (see [**8**, Theorem 5.5]); the most important of these for our purposes is the following.

Let $I \subset S$. If $w = s_{i_1} \cdots s_{i_r}$ ($s_i \in S$) is a reduced expression, and $w \in W_I$, then all $s_i \in I$. In particular, the function ℓ agrees with the length function ℓ_I on W_I, and $W_I \cap S = I$.

We say that a typical basis element $u_{\mathbf{i},\mathbf{j}}$ of $T^r(M_n)$ is of *parabolic form* if the stabilizer of $\mathbf{j}$ under the action of $\mathcal{S}_r$ on $I(n,r)$ is a parabolic subgroup. We will also say in this case that the element $\mathbf{j}$ is of parabolic form. An equivalent way of looking at this is that if $1 \leq a < b < c \leq r$ and $j_a = j_c$, then $j_a = j_b = j_c$.

1.8 Two orders

Let A and B be ordered sets. Then the *Hebrew lexicographic order* on $A \times B$ is given by $\langle a, b\rangle < \langle c, d\rangle$ if and only if $b < d$ or ($b = d$ and $a < c$). The elements of a sequence $(i_1, \ldots, i_m)$ in $A \times B$ are said to appear in Hebrew lexicographic order if $i_1 \leq \cdots \leq i_r$ with respect to this order.

A basis element $u_{\mathbf{i},\mathbf{j}}$ of $T^r(M_n)$ is said to be *H-relaxed* if the elements of the sequence $(\langle i_1, j_1\rangle, \ldots, \langle i_r, j_r\rangle)$ appear in Hebrew-lexicographic order.

Let A and B be ordered sets. Then the *relaxed order* on $A \times B$ is given by $\langle a, b\rangle < \langle c, d\rangle$ if and only if $b < d$ or ($b = d$ and $a > c$). The elements of a sequence $(i_1, \ldots, i_m)$ in $A \times B$ are said to appear in relaxed order if $i_1 \leq \cdots \leq i_r$ with respect to this order.

A basis element $u_{\mathbf{i},\mathbf{j}}$ of $T^r(M_n)$ is said to be *relaxed* if the elements of the sequence $(\langle i_1, j_1\rangle, \ldots, \langle i_r, j_r\rangle)$ appear in the relaxed order.

Examples

Let $\mathbf{j} = (1, 1, 1, 1, 3, 3, 3, 7, 7, 7)$.

If $\mathbf{i} = (1, 4, 6, 7, 2, 4, 5, 3, 3, 4)$ then $u_{\mathbf{i},\mathbf{j}}$ is H-relaxed.

If $\mathbf{i}' = (7, 6, 4, 1, 5, 4, 2, 4, 3, 3)$ then $u_{\mathbf{i}',\mathbf{j}}$ is relaxed.

Note

Under the canonical identifications, these definitions of relaxed and H-relaxed agree with those in [**7**, §3].

1.9 Quantized codeterminants and combinatoric definitions

A *quantized codeterminant*, or *q-codeterminant*, is an element $e_A e_{A'}$ of the q-Schur algebra $S_q(n, r)$ which is a nonzero product of two natural basis elements e_A and $e_{A'}$. (The quantized codeterminant is determined by the matrices A and A', and not solely by the element of $S_q(n, r)$ which is equal to it.)

The above discussion shows that this definition specialises (when q is replaced by 1) to the classical definition of codeterminants. We will show in this paper that the quantized codeterminants as defined above exhibit some of the important properties of their classical counterparts.

In keeping with the classical literature on codeterminants, we will tend to follow the notation of [**6**]. We therefore need to make a few more definitions, many analogous to the ones in [**6**], before motivating the main results of this paper.

It is convenient to introduce

$$\Lambda = \Lambda(n, r) := \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_\nu \in \mathbf{N}_0 \text{ for all } \nu \in \mathbf{n}, \text{ and } \sum_{\nu=1}^{n} \lambda_\nu = r \right\}$$

and

$$\Lambda^+ = \Lambda^+(n, r) := \{\lambda \in \Lambda(n, r) : \lambda_1 \geq \cdots \geq \lambda_n\}.$$

(Here, $\mathbf{N}_0$ is the set of natural numbers, including 0.)

An element λ of Λ is called a *weight*, and is *dominant* if $\lambda \in \Lambda^+$. There is an obvious correspondence between elements of Λ^+ and partitions of r into not more than n parts.

The lexicographic order on Λ is defined as follows. $\mu \succ \lambda$ if for some c, $\mu_a = \lambda_a$ for all $a < c$, and $\mu_c > \lambda_c$.

The weight $\mathrm{wt}(\mathbf{i})$ of an element $\mathbf{i} \in I(n, r)$ is the element $\alpha \in \Lambda$ given by $\alpha_\nu = |\{\rho \in \mathbf{r} : i_\rho = \nu\}|$ for all $\nu \in \mathbf{n}$. If $\mathbf{i}, \mathbf{j} \in I$, it is clear that $\mathbf{i} \sim \mathbf{j}$ if and only if $\mathrm{wt}(\mathbf{i}) = \mathrm{wt}(\mathbf{j})$.

For each $\lambda \in \Lambda^+$ we define the *basic λ-tableau* T^λ by writing the integers $1, \ldots, r$ into a Young diagram row by row, starting with the top row, and filling each row from left to right. To each $\mathbf{i} \in I$ we now associate the λ-tableau $T_{\mathbf{i}}^\lambda = \mathbf{i}T^\lambda$. For example, let $n = 4$, $r = 7$, $\lambda = (4, 2, 1, 0)$. Then

$$T^\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \; ,$$

and

$$T_{\mathbf{i}}^{\lambda} = \begin{array}{|c|c|c|c|} \hline i_1 & i_2 & i_3 & i_4 \\ \hline i_5 & i_6 \\ \cline{1-2} i_7 \\ \cline{1-1} \end{array} \, .$$

If $\lambda \in \Lambda^+$ and $\mathbf{i} \in I$, the λ-tableau $T_{\mathbf{i}}^{\lambda}$ is said to be *standard* if the elements in each row increase weakly from left to right, and the elements in each column increase strictly from top to bottom. We define

$$I_\lambda := I_\lambda(n,r) = \{\mathbf{i} \in I : T_{\mathbf{i}}^{\lambda} \text{ is standard}\}.$$

We say a λ-tableau $T_{\mathbf{i}}^{\lambda}$ is *row-semistandard* if the elements in each row increase weakly from left to right, and define the set

$$I'_\lambda := I'_\lambda(n,r) = \{\mathbf{i} \in I : T_{\mathbf{i}}^{\lambda} \text{ is row-semistandard}\}.$$

It is clear that in a standard tableau, all the entries equal to s must appear in the first s rows, by an easy induction on s. There is exactly one element of I, denoted by $\ell = \ell(\lambda)$, for which T_{ℓ}^{λ} is standard and $\mathrm{wt}(\ell) = \lambda$. (The entries in the s-th row of the tableau T_{ℓ}^{λ} are all equal to s.)

The elements of I can be lexicographically ordered as follows: $\mathbf{i} \succeq \mathbf{j}$ if and only if $\mathbf{i} = \mathbf{j}$ or, if ρ is minimal subject to the condition that $i_\rho \neq j_\rho$, then $i_\rho < j_\rho$.

It is clear for any $\mathbf{p} \in I$ that $\mathbf{p} \sim \ell(\mathrm{wt}(\mathbf{p}))$. Thus, any basis element $\xi_{\mathbf{a},\mathbf{b}}$ can be written as $\xi_{\mathbf{i},\ell(\mu)}$ where $\mu = \mathrm{wt}(\mathbf{b})$, or as $\xi_{\ell(\nu),\mathbf{j}}$ where $\nu = \mathrm{wt}(\mathbf{a})$. The product $\xi_{\mathbf{a},\mathbf{b}}\xi_{\mathbf{c},\mathbf{d}}$ will be zero unless $\mathbf{b} \sim \mathbf{c}$, and hence $\mathrm{wt}(\mathbf{b}) = \mathrm{wt}(\mathbf{c})$, so it is easy to see that a general codeterminant can be written as

$$Y_{\mathbf{i},\mathbf{j}}^{\lambda} := \xi_{\mathbf{i},\ell(\lambda)}\xi_{\ell(\lambda),\mathbf{j}},$$

where $\mathbf{i},\mathbf{j} \in I$ and $\lambda \in \Lambda$. We say that $Y_{\mathbf{i},\mathbf{j}}^{\lambda}$ is a codeterminant *of shape* λ. If, in addition, $\lambda \in \Lambda^+$ and $\mathbf{i}$ and $\mathbf{j}$ lie in I_λ, we call the codeterminant $Y_{\mathbf{i},\mathbf{j}}^{\lambda}$ a *standard* codeterminant. It should be noted that if $\mathbf{b} \sim \mathbf{c}$ then $\xi_{\mathbf{a},\mathbf{b}}\xi_{\mathbf{c},\mathbf{d}}$ is nonzero, because it specialises when q is replaced by 1 to an element of $S(n,r)$ which can be seen to be nonzero using Schur's product rule.

It is convenient to introduce the following elements of $\mathbf{Z}[q]$, as in [**2**].

a) For an nonnegative integer n, $[n]_q := 1 + q + \cdots + q^{n-1}$.

b) Define $[n]_q! := [1]_q[2]_q \cdots [n]_q$.

Let $\mathbf{a}, \mathbf{b} \in I(n,r)$. Then the element $P_{\mathbf{a},\mathbf{b}} \in \mathbf{Z}[q]$ is defined to be

$$\prod_{i,j=1}^{n} [c_{ij}]_q!,$$

where c_{ij} is the multiplicity of $\langle i,j \rangle$ in the family $\{\langle a_1, b_1 \rangle, \dots, \langle a_r, b_r \rangle\}$. If $t = u_{\mathbf{a},\mathbf{b}}$ then $P_t := P_{\mathbf{a},\mathbf{b}}$.

1.10 Computation of products in $S_v(n,r)$

In [**7**, §3], it was shown how to embed the algebra $S_v(n,r)$ into $T^r(M_n)$, and, since $T^r(M_n)$ has a convenient multiplication defined on it, this was shown to give an algorithm for computing products in $S_v(n,r)$. We summarise the most important results in [**7**] for use in this paper, although the notation differs somewhat.

Let A be a basis matrix for $S_v(n,r)$. Define the integer

$$m_A := \sum_{i,j,k} A_{i,k} A_{j,k},$$

where the sum is taken over all $i,j,k \in \{1, \dots, n\}$ subject to the condition $i < j$. Define the integer

$$s_A := \sum_{1 \leq i,j,k,l \leq n} A_{i,j} A_{k,l},$$

where the sum is taken with the restrictions that $i \geq k$ and $j < l$.

Proposition 1.2 Let W denote the Coxeter group of type A_{r-1}. Suppose $u_{\mathbf{i},\mathbf{j}}$ is H-relaxed. Let $(\mathbf{i}', \mathbf{j})$ be the unique relaxed element satisfying

$$(\mathbf{i}, \mathbf{j}) \sim (\mathbf{i}', \mathbf{j}).$$

(i) The linear map $\gamma : S_v(n,r) \to T^r(M_n)$ given by

$$\gamma(\xi_{\mathbf{i},\mathbf{j}}) := v^{s_A - m_A} u_{\mathbf{i},\mathbf{j}} . \frac{x_W}{P_{\mathbf{i},\mathbf{j}}}$$

is a monomorphism of algebras. (It should be noted there are many ways of expressing a basis element $\xi_{\mathbf{i},\mathbf{j}}$, and that this formula is only valid when the subscripts $\mathbf{i}$ and $\mathbf{j}$ occurring satisfy the condition that $u_{\mathbf{i},\mathbf{j}}$ is H-relaxed. Each basis element can be expressed in this form in a unique way.)

(ii) $\mathrm{im}(\gamma)$ coincides with the largest $\mathcal{H}$-submodule of $T^r(M_n)$ affording the ind representation of $\mathcal{H}$ (i.e. the representation mapping all T_{s_i} to q.) We shall later refer to this submodule as the *ind submodule.* Working over the field $\mathbf{Q}(v)$, this submodule is given by $T^r(M_n).x_W$.

(iii) Let $\xi_{\mathbf{i},\mathbf{j}} = e_A$. The coefficient of $u_{\mathbf{i},\mathbf{j}}$ in $\gamma(e_A)$ is $v^{s_A - m_A}$. The coefficient of $u_{\mathbf{i}',\mathbf{j}}$ in $\gamma(e_A)$ is v^{s_A}. There are no other relaxed or H-relaxed basis elements which appear with nonzero coefficients in $\gamma(e_A)$.

Proof The elements $[A] \in S_v(n,r)$ used in [**7**] satisfy $e_A = v^{s_A}[A]$, as was shown in [**7**, §3].

The assertions of (i) now follow from [**7**, Theorem 3.9].

The assertions of (ii) follow from [**7**, Lemma 3.8 (a)].

For the proof of (iii), see [**7**, Corollary 3.7]. ∎

In some of the proofs in this paper, we will use the technique of proving that an element $w \in W$ satisfies a certain property P by "induction on left subwords of w." The idea is as follows. Let $s_{i_1} \cdots s_{i_m}$ be any fixed reduced expression for w. We then prove by induction on p (where $1 \leq p \leq m$) that the element w_p defined by $s_{i_1} \cdots s_{i_p}$ satisfies the property P; this then proves that w has the required property. It is immediate from the properties of Coxeter groups that if w is a right coset representative of minimial length with respect to some parabolic subgroup of W, then so are all its left subwords. It is also immediate that if w lies in some parabolic subgroup W_P of W, then so do all its left subwords.

Proposition 1.3

Suppose that $j_a \leq j_b$ when $a < b$. Then $\mathbf{j}$ is of parabolic form, with stabiliser W_J, and

(i) If $u_{\mathbf{a},\mathbf{j}}$ occurs with nonzero coefficient in $u_{\mathbf{i},\mathbf{j}}.T_w$ for some w then $(\mathbf{a},\mathbf{j}) \sim (\mathbf{i},\mathbf{j})$.

(ii) If $w \notin W_J$, then the coefficient of $u_{\mathbf{a},\mathbf{j}}$ in $u_{\mathbf{i},\mathbf{j}}.T_w$ is zero for all $\mathbf{a} \in I(n,r)$.

Now suppose $u_{\mathbf{i},\mathbf{j}}$ is H-relaxed, and let W_P be the parabolic subgroup which is the stabiliser of $(\mathbf{i},\mathbf{j})$. Let W_J be the parabolic subgroup which is the stabiliser of $\mathbf{j}$. Let e_A be the basis element of $S_q(n,r)$ corresponding to $\xi_{\mathbf{i},\mathbf{j}}$. The following two remarks now hold.

(iii) Suppose $(\mathbf{a},\mathbf{j}) \sim (\mathbf{i},\mathbf{j})$. Then the coefficient of $u_{\mathbf{a},\mathbf{j}}$ in $\gamma(\xi_{\mathbf{i},\mathbf{j}})$ is $v^{s_A - m}$ where

$$m := |\{\langle c,d\rangle : j_c = j_d, a_c < a_d\}|.$$

(iv) Suppose $w \in \mathcal{D}_P \cap W_J$. Then $u_{\mathbf{i},\mathbf{j}}.T_w = v^{\ell(w)} u_{\mathbf{i}.w,\mathbf{j}.w}$.

Proof

Observe that any basis element occurring with nonzero coefficient in $u_{\mathbf{i},\mathbf{j}}.T_w$ is of form $u_{\mathbf{a},\mathbf{j}.w}$. This follows by induction on left subwords of w. Also observe that if $w \in W_J$ then $u_{\mathbf{i},\mathbf{j}}.T_w$ is of form $u_{\mathbf{a},\mathbf{j}}$ where $(\mathbf{a},\mathbf{j}) \sim (\mathbf{i},\mathbf{j})$, also by induction on left subwords of w. This is enough to prove (i) and (ii).

The assertions (iii) and (iv) follow from [**7**, Lemma 3.6] and its proof. ∎

2. Hecke algebras and codeterminants of dominant shape

Lemma 2.1 Let $t = u_{\mathbf{a},\mathbf{b}}$ (where $\mathbf{a}, \mathbf{b} \in I(n,r)$) be a basis element of $T^r(M_n)$ of parabolic form. Then

$$y := t.\frac{x_W}{P_t}$$

lies in $S_v(n,r)$. Moreover, if t is expressed as an $\mathcal{A}$-linear sum of elements $\xi_{\mathbf{i},\mathbf{j}}$, then all elements occurring with nonzero coefficients in the sum satisfy $\mathbf{i} \sim \mathbf{a}$ and $\mathbf{j} \sim \mathbf{b}$.

Proof Lemma 1.1 shows that

$$T_d x_W = q^{\ell(d)} x_W,$$

where W is the Coxeter group of type A_{r-1}. We now find that y lies in the ind submodule of $T^r(M_n)$, and hence in $S_v(n,r)$ by Proposition 1.2 (ii). This proves that t is a $\mathbf{Q}(v)$-linear combination of elements $\xi_{\mathbf{i},\mathbf{j}}$. Thanks to Proposition 1.2 (iii), it is now enough to show that the coefficients of the relaxed basis elements occurring in y lie in $\mathcal{A}$, because that will force the coefficient of each $\xi_{\mathbf{i},\mathbf{j}}$ to lie in $\mathcal{A}$.

Define W_J to be the parabolic subgroup of W generated by the $(p, p+1)$ satisfying $b_p = b_{p+1}$. Let t_0 be the unique element of form $u_{\mathbf{a}',\mathbf{b}}$ satisfying $(\mathbf{a}', \mathbf{b}) \sim (\mathbf{a}, \mathbf{b})$ and the condition that if $b_p = b_{p+1}$ then $a'_p \leq a'_{p+1}$. Let W_I be the parabolic subgroup of $\mathcal{S}_r$ stabilising t_0. There exists a unique $w \in \mathcal{D}_I \cap W_J$ such that $\mathbf{a}'.w = \mathbf{a}$. Then we see by induction on left subwords of w that $u_{\mathbf{a}',\mathbf{b}}.T_w = v^* u_{\mathbf{a},\mathbf{b}}$, as required. By substituting t_0 for t in the definition of y, and by using the observation that if $a \in \mathcal{A}$ then $v^m a \in \mathcal{A}$ for any $m \in \mathbf{Z}$, we find that it is enough to prove the lemma in the case where $t = t_0$, because $P_t = P_{t_0}$.

We now assume $t = t_0$ and define the parabolic subgroup W_I to be generated by all $(p, p+1)$ such that $a'_p = a'_{p+1}$ and $b_p = b_{p+1}$. Let $\mathcal{D}_I$ be the set of distinguished right coset representatives corresponding to W_I, that is, the set

$$\mathcal{D}_I = \{w \in W : \ell(sw) > \ell(w) \text{ for all } s \in I\}.$$

By using a reversed version of [**8**, Proposition 1.10], we find that for any $w \in W$, there exists a unique $a \in W_I$ and $b \in \mathcal{D}_I$ such that $w = ab$ and $\ell(w) = \ell(a) + \ell(b)$. Each element of $\mathcal{D}_I$ is the unique element of minimal length in its right coset. It follows that

$$x_W = x_I \left(\sum_{u \in \mathcal{D}_I} T_u \right).$$

The Poincaré polynomial $W(z)$ associated to the Coxeter system (W, S) is defined to be

$$W(z) := \sum_{n \geq 0} a_n z^n = \sum_{w \in W} z^{\ell(w)},$$

where $a_n := |\{w \in W : \ell(w) = n\}|$. We know (see for example, [**8**, §3.15]) that the Poincaré polynomial of a Coxeter group of type A_l is

$$\prod_{i=1}^{l} \sum_{k=0}^{i} z^k,$$

and we also know that W_I, being a parabolic subgroup of a Coxeter group of type A, is canonically isomorphic to a direct product of such groups of smaller rank. Its Poincaré polynomial is therefore a product of expressions such as those given above. By substituting $z = q$, we obtain the scalar by which $\sum_{u \in W_I} T_u$ acts, and we find that

$$t.x_I = P_t.t.$$

This means that

$$y = t. \left(\sum_{u \in \mathcal{D}_I} T_u \right),$$

and hence, since the action of $\mathcal{H}$ on $T^r(M_n)$ is defined over $\mathcal{A}$, that the coefficients of any basis elements occurring in y, in particular the relaxed ones, lie in $\mathcal{A}$.

The last assertion follows from Proposition 1.2 (iii) and the Note at the end of §1.6. ∎

Definition Let $\mathbf{i}, \mathbf{j} \in I(n,r)$ be given by $\mathbf{i} = (i_1, \ldots, i_r)$ and $\mathbf{j} = (j_1, \ldots, j_r)$. Then the element $\tau_{\mathbf{i},\mathbf{j}} \in T^r(M_n)$ is defined as

$$\tau_{\mathbf{i},\mathbf{j}} := u_{\mathbf{i},\mathbf{j}} . \frac{x_W}{P_{\mathbf{i},\mathbf{j}}}.$$

If $\mathbf{j}$ is of parabolic form, we know from Lemma 2.1 that $\tau_{\mathbf{i},\mathbf{j}}$ can be considered to lie in $S_v(n,r)$.

Note that if the $u_{\mathbf{i},\mathbf{j}}$ constructed above is H-relaxed, then $\tau_{\mathbf{i},\mathbf{j}}$ is equal to $\xi_{\mathbf{i},\mathbf{j}}$ multiplied by a certain power of v.

Also note that Lemma 2.1 tells us that each element $\tau_{\mathbf{i},\mathbf{j}}$ is of form

$$\sum c_{\mathbf{i}',\mathbf{j}'} \xi_{\mathbf{i}',\mathbf{j}'},$$

where the elements $c_{\mathbf{i}',\mathbf{j}'}$ lie in $\mathcal{A}$, and the indices $\mathbf{i}'$ and $\mathbf{j}'$ satisfy $\mathbf{i}' \sim \mathbf{i}$ and $\mathbf{j}' \sim \mathbf{j}$.

During the proofs that follow, we shall need a way of testing whether two elements of $\mathcal{H}$ are equal to each other. To do this, we construct a right $\mathcal{H}$-module M with the property that there exist certain elements $m \in M$ such that if $h_1, h_2 \in \mathcal{H}$ and $m.h_1 = m.h_2$ then $h_1 = h_2$. By linearity, it is enough to show that if $m.h = 0$ then $h = 0$.

Lemma 2.2 Let V be a $\mathbf{Q}(v)$-vector space of dimension r with basis $e_1, \ldots, e_r$. We make $M := V^{\otimes r}$ into a right $\mathcal{H}$-module in the usual way (i.e. via the tensor space action defined in 1.5). Let $h \in \mathcal{H}$. Then:

(i) if $m := e_1 \otimes \cdots \otimes e_r$, then $m.h = 0 \Rightarrow h = 0$;

(ii) if $m_\pi := e_{(1)\pi} \otimes \cdots \otimes e_{(r)\pi}$, where $\pi \in \mathcal{S}_r$, then $m_\pi.h = 0 \Rightarrow h = 0$.

Proof We first prove (i). Let m and m_π be defined as in the statement of the Lemma. By using induction on $\ell(\pi)$, we find that $m.T_\pi = v^{\ell(\pi)} m_\pi$. If $\pi \neq \pi'$ then $m_\pi \neq m_{\pi'}$; in fact, the set $\{m_\pi : \pi \in \mathcal{S}_r\}$ is linearly independent in $V^{\otimes r}$. However, $\{T_w : w \in \mathcal{S}_r\}$ is known to be a basis for $\mathcal{H}$. Since v is invertible, we now find that if $m.h = 0$, then $h = 0$.

We now prove (ii). We know from the above argument that $m.T_\pi = v^{\ell(\pi)} m_\pi$. Assuming that $m_\pi.h = 0$, then $m.(v^{-\ell(\pi)} T_\pi h) = 0$. Now (i) shows that $v^{-\ell(\pi)} T_\pi h = 0$, and, since v and T_π are both invertible, that $h = 0$. This completes the proof. ∎

We now need to introduce a family of bases for $\mathcal{H}$, of which the usual basis $\{T_w\}$ is an example. The motivation behind doing this is that by picking the correct basis, it is possible to deduce certain results concerning the coefficients of basis elements in the $\tau_{\mathbf{i},\mathbf{j}}$ which were introduced earlier. This will be important later.

For convenience, we now choose, once and for all, a "favourite" reduced expression $\rho(w)$ for each element $w \in W$.

Definition Let $\mathbf{j} \in I(n,r)$ be of parabolic form, let W_J be its associated parabolic subgroup, and let $\mathcal{D}_J$ be the associated set of distinguished right coset representatives. To each element $w \in \mathcal{D}_J$, we will associate an element $D_w^{\mathbf{j}} \in \mathcal{H}$ which specialises as q is replaced by 1 to w in the group algebra.

Suppose $\rho(w) = s_{i_1} \cdots s_{i_m}$. Earlier, we showed how $\mathcal{S}_r$ acts on $I(n,r)$ on the right. For each integer t such that $1 \leq t \leq m$ we define an element h_t of $\mathcal{H}$ as follows. Consider $\mathbf{j}' := \mathbf{j}.s_{i_1} \cdots s_{i_{t-1}}$. Because w is a distinguished right coset representative, it must be the case that $j'_{i_t} \neq j'_{i_{t+1}}$. (This follows from the length identities in 1.4, the definition of distinguished right coset representatives, and the fact that any left subword of a distinguished right coset representative is another distinguished right coset representative.) Writing T_{i_t} instead of $T_{s_{i_t}}$ (for typographical convenience), we find that if $j'_{i_t} > j'_{i_{t+1}}$ then $h_t := T_{i_t}^{-1}$, otherwise $h_t := T_{i_t}$. We then define $D_w^{\mathbf{j}}$ to be $h_1 h_2 \cdots h_m$.

Remark

If $1 \leq a < b \leq n$, then $e_b \otimes e_a.T_s^{-1} = v^{-1} e_a \otimes e_b$ and $e_a \otimes e_b.T_s = v e_b \otimes e_a$. Define, for any $\mathbf{i} \in I(n,r)$,

$$m(\mathbf{i}) := |\{\langle a, b\rangle : 1 \leq a < b \leq r,\ i_a > i_b\}|.$$

(For example, if $i = ((1)\pi, \ldots, (r)\pi)$ as in the statement of Lemma 2.2 (ii), then $m(\mathbf{i}) = \ell(\pi)$.)

If $s = (p, p+1)$ and $j_p \leq j_{p+1}$ then $u_{\mathbf{i},\mathbf{j}}.T_s$ is a linear combination of elements $u_{\mathbf{i}',\mathbf{j}.s}$ (see §1.6, (1) and (3)). If $j_p \geq j_{p+1}$ then $u_{\mathbf{i},\mathbf{j}}.T_s^{-1}$ is a linear combination of elements $u_{\mathbf{i}',\mathbf{j}.s}$ (see §1.6, (5) and (7)).

The crucial property of the element $D_w^{\mathbf{j}}$ is as follows:

$$v^{m(\mathbf{j})} e_{j_1} \otimes \cdots \otimes e_{j_r}.D_w^{\mathbf{j}} = v^{m(\mathbf{j}.w)} e_{(j_1)w} \otimes \cdots \otimes e_{(j_r)w}.$$

This can be seen by induction on left subwords of w.

It is also follows that basis elements occurring with nonzero coefficients in $u_{\mathbf{i},\mathbf{j}}.D^{\mathbf{j}}_w$ are of form $u_{\mathbf{i}',\mathbf{j}.w}$.

Example

Let $\mathbf{j} = (4,4,1,2)$. Then the parabolic subgroup W_J is generated by $s_1 = (1,2)$. A routine check shows that the element $w \in \mathcal{S}_4$ given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

lies in $\mathcal{D}_J$. The element $\mathbf{j}.w \in I$ is equal to $(2,4,1,4)$. A reduced expression $\rho(w)$ for w is given by $s_3 s_2 s_1 s_3$. The element $D^{\mathbf{j}}_w \in \mathcal{H}$ is now given by $T_3 T_2^{-1} T_1^{-1} T_3^{-1}$. (It will be seen later that the element $D^{\mathbf{j}}_w$ is independent of the reduced expression chosen for w.)

We now compare the elements $D^{\mathbf{j}}_w$ with their counterparts T_w. To do this, it is convenient to define an element $R(\mathbf{j}) \in I(r,r)$ (N.B.—not necessarily in $I(n,r)$) which is in some sense a "refinement" of $\mathbf{j}$. We will then act various $D^{\mathbf{j}}_w$ on this $R(\mathbf{j})$, and appeal to Lemma 2.2 to relate them to the usual basis consisting of the T_w.

Definition Let $\mathbf{i} \in I(n,r)$ be of parabolic form. Then the element $R(\mathbf{i})$ is the unique element of $I(r,r)$ satisfying the following properties:

(i) $R(\mathbf{i}) = ((1)\pi, \dots, (r)\pi)$ for some $\pi \in \mathcal{S}_r$;

(ii) if $1 \leq a < b \leq r$ and $i_a < i_b$ (respectively $i_a > i_b$) then $R(\mathbf{i})_a < R(\mathbf{i})_b$ (respectively $R(\mathbf{i})_a > R(\mathbf{i})_b$);

(iii) if $1 \leq a < b \leq r$ and $i_a = i_b$ then $R(\mathbf{i})_a < R(\mathbf{i})_b$.

Example Let $n = 4, r = 8$ and $\mathbf{i} = (1,1,1,4,4,4,3,3)$. Then $R(\mathbf{i}) = (1,2,3,6,7,8,4,5)$.

Remarks Note that $m(R(\mathbf{i})) = m(\mathbf{i})$ for any $\mathbf{i} \in I(n,r)$ of parabolic form. Also note that

$$v^{m(R(\mathbf{i}))} e_{R(\mathbf{i})_1} \otimes \cdots \otimes e_{R(\mathbf{i})_r} . D^{\mathbf{i}}_w = v^{m(R(\mathbf{i}).w)} e_{R(\mathbf{i})_{(1)w}} \otimes \cdots \otimes e_{R(\mathbf{i})_{(r)w}} .$$

Together with Lemma 2.2, this shows that the element $D^{\mathbf{j}}_w$ is in fact independent of the reduced expression we were using for w.

We can say more than this: if W_I is the parabolic subgroup associated with $\mathbf{i}$, and $u \in W_I$, then

$$v^{m(R(\mathbf{i}))} e_{R(\mathbf{i})_1} \otimes \cdots \otimes e_{R(\mathbf{i})_r} . T_u D^{\mathbf{i}}_w = v^{m(R(\mathbf{i}).uw)} e_{R(\mathbf{i})_{(1)uw}} \otimes \cdots \otimes e_{R(\mathbf{i})_{(r)uw}} .$$

This follows from the definitions of $R(\mathbf{i})$ and $D^{\mathbf{i}}_w$: all the actions are essentially permutations, but with powers of v introduced to keep track of the changing value of the m function (which can be thought of as "length").

Lemma 2.3 Let $\mathbf{j} \in I(n,r)$ be of parabolic form, with associated parabolic subgroup W_J. Then:

(i)

$$\left(\sum_{u \in W_J} T_u\right)\left(\sum_{w \in \mathcal{D}_J} D^{\mathbf{j}}_w\right) = q^{-m(\mathbf{j})} \sum_{w \in W} T_w.$$

(ii) The set

$$\{T_u D^{\mathbf{j}}_w : u \in W_J, w \in \mathcal{D}_J\}$$

is a basis for $\mathcal{H}$.

Proof Let $\pi \in \mathcal{S}_r$ be such that $R(\mathbf{j}) = ((1)\pi, \ldots, (r)\pi)$. Then

$$e_1 \otimes \cdots \otimes e_r.T_\pi = v^{m(R(\mathbf{j}))} e_{R(\mathbf{j})_1} \otimes \cdots \otimes e_{R(\mathbf{j})_r}.$$

It now follows that

$$v^{m(R(\mathbf{j}))} e_{R(\mathbf{j})_1} \otimes \cdots \otimes e_{R(\mathbf{j})_r}.T_\pi^{-1}.\left(\sum_{w \in W} T_w\right) = \sum_{w \in W} v^{\ell(w)} e_{(1)w} \otimes \cdots \otimes e_{(r)w}.$$

It also follows from the preceding remarks that

$$v^{m(R(\mathbf{j}))} e_{R(\mathbf{j})_1} \otimes \cdots \otimes e_{R(\mathbf{j})_r}.\left(\sum_{u \in W_J} T_u\right)\left(\sum_{w \in \mathcal{D}_J} D^{\mathbf{j}}_w\right) = \sum_{w \in W} v^{\ell(w)} e_{(1)w} \otimes \cdots \otimes e_{(r)w}.$$

Applying Lemma 2.2 (ii), this shows that

$$\left(\sum_{u \in W_J} T_u\right)\left(\sum_{w \in \mathcal{D}_J} D^{\mathbf{j}}_w\right) = T_\pi^{-1}.\left(\sum_{w \in W} T_w\right),$$

where the right-hand side is known to be equal to

$$q^{-\ell(\pi)} \sum_{w \in W} T_w,$$

thus establishing (i) since $\ell(\pi) = m(R(\mathbf{j})) = m(\mathbf{j})$.

Part (ii) follows easily from the definitions and the Bruhat order relation, because $D^{\mathbf{j}}_w$ is a linear combination of T_w and T_y with $y < w$. ∎

Definition Let $\mathbf{z} \in I(n,r)$. Define $\mathbf{z}^+$ to be the element of $I(n,r)$ satisfying $z_i^+ = f(z_i)$ for all i and for $f \in \mathcal{S}_n$ of minimal length subject to the condition that $\mathrm{wt}(\mathbf{z}^+) \in \Lambda^+(n,r)$. (Such an f is uniquely defined, as will be illustrated in the following example.)

Example Let $n = 5$, $r = 9$ and $\mathbf{z} = (1,1,2,2,2,2,4,4,4)$, so that $\mathrm{wt}(\mathbf{z}) = (2,4,0,3,0)$. Then the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

works in the above definition of f, and

$$\mathbf{z}^+ = (3,3,1,1,1,1,2,2,2),$$

with

$$\mathrm{wt}(\mathbf{z}^+) = (4,3,2,0,0).$$

Note that $\mathrm{wt}(\mathbf{z})$ is a permutation of $\mathrm{wt}(\mathbf{z}^+)$, and that $\mathrm{wt}(\mathbf{z}) \in \Lambda^+$. This will always be the case, and in fact $\mathrm{wt}(\mathbf{z}^+) = f.\mathrm{wt}(\mathbf{z})$. There is exactly one such f of minimal length, showing why f and $\mathbf{z}^+$ are well-defined.

Lemma 2.4 Let $\mathbf{z} \in I(n,r)$ be such that $i < j \Rightarrow z_i \leq z_j$. In particular, $\mathbf{z}$ is of parabolic form; let the associated parabolic subgroup be W_Z. Then:

(i) $\tau_{\mathbf{z},\mathbf{z}^+}\tau_{\mathbf{z}^+,\mathbf{z}} = v^*\tau_{\mathbf{z},\mathbf{z}}$;

(ii) There exists $u \in S_v(n,r)$ satisfying $u.\xi_{\mathbf{z}^+,\mathbf{z}} = v^*\xi_{\mathbf{z}}$.

Proof It is immediate that if $\mathbf{z}$ is of parabolic form, then so is $\mathbf{z}^+$, and the stabiliser of $\mathbf{z}^+$ is precisely W_Z. We then know from the remarks following the definition of the $\tau_{\mathbf{i},\mathbf{j}}$ that $\tau_{\mathbf{z}^+,\mathbf{z}} = v^*\xi_{\mathbf{z}^+,\mathbf{z}}$, for some integer $*$, and that $\tau_{\mathbf{z},\mathbf{z}} = v^*\xi_{\mathbf{z},\mathbf{z}}$ for some integer $*$. Furthermore, we know that $\tau_{\mathbf{z},\mathbf{z}^+}$ corresponds to some element in $S_v(n,r)$. This establishes the proof of (ii) assuming (i), so it remains to prove (i).

To prove (i), it is enough, by the results of Proposition 1.2, to prove that the only relaxed basis element occurring in $\tau_{\mathbf{z},\mathbf{z}^+}\tau_{\mathbf{z}^+,\mathbf{z}}$ is that corresponding to $\tau_{\mathbf{z},\mathbf{z}}$, and that it occurs with a coefficient which is a power of v.

The only relaxed basis element occurring with nonzero coefficient in $\tau_{\mathbf{z}^+,\mathbf{z}}$ which can contribute to a relaxed basis element in the product is $u_{\mathbf{z}^+,\mathbf{z}}$. This follows from Proposition 1.3 and the fact that $\tau_{\mathbf{z}^+,\mathbf{z}} = v^*.\xi_{\mathbf{z}^+,\mathbf{z}}$. Proposition 1.3 (iii) tells us that the coefficient of $u_{\mathbf{z}^+,\mathbf{z}}$ is a power of v.

The problem now reduces further—it is now enough to show that if $u_{\mathbf{a},\mathbf{z}^+}$ occurs in $\tau_{\mathbf{z},\mathbf{z}^+}$ with nonzero coefficient then $\mathbf{a} = \mathbf{z}$ and the coefficient is a power of v.

Using Lemma 2.3 (i), the definition of τ, and the action of T_y (where $y \in W_Z$) on $u_{\mathbf{z},\mathbf{z}^+}$, we find that

$$\tau_{\mathbf{z},\mathbf{z}^+} = v^*.u_{\mathbf{z},\mathbf{z}^+}.\left(\sum_{w \in \mathcal{D}_Z} D_w^{\mathbf{z}^+}\right).$$

From our study of the properties of $D_w^{\mathbf{z}^+}$, we see that any basis element occurring with nonzero coefficient in $u_{\mathbf{z},\mathbf{z}^+} D_w^{\mathbf{z}^+}$, where w is as in the above sum, is of form $u_{\mathbf{b},\mathbf{z}^+.w}$. The implication of this is that the only basis elements of the desired form come from the case $w = 1$ in the sum. This is enough to complete the proof of the Lemma. ∎

Remark on Lemma 2.4 In the classical case, this result is obvious (see [**9**]).

Proposition 2.5 Any codeterminant is a $\mathbf{Z}[q, q^{-1}]$-linear combination of codeterminants of dominant shape.

Proof We know that $u.\xi_{\mathbf{z}^+,\mathbf{z}} = \xi_{\mathbf{z}}$ for some $u \in S_v(n, r)$, because v is a unit. Since

$$S_v(n, r) \cong S_q(n, r) \oplus v.S_q(n, r)$$

as a $\mathbf{Z}[q, q^{-1}]$ module, we can take u to be in $S_q(n, r)$, by omitting irrelevant terms.

Since $u.\xi_{\mathbf{z}^+,\mathbf{z}} = u.\xi_{\mathbf{z}^+}.\xi_{\mathbf{z}^+,\mathbf{z}}$, we see that $u.\xi_{\mathbf{z}^+}$ is a $\mathbf{Z}[q, q^{-1}]$-linear combination of basis elements of the form $\xi_{\mathbf{a},\mathbf{z}^+}$. It now follows that $\xi_{\mathbf{z}}$ is a $\mathbf{Z}[q, q^{-1}]$-linear combination of codeterminants of shape $\mathbf{z}^+$, i.e. of dominant shape.

Now consider an arbitrary codeterminant, $Y_{\mathbf{i},\mathbf{j}}^{\mathrm{wt}(\mathbf{z})}$, where $\mathbf{z}$ can be taken to be as in the statement of Lemma 2.4 (and so in particular, $\mathbf{z} = \ell(\mathrm{wt}(\mathbf{z}))$). This is equal to $\xi_{\mathbf{i},\mathbf{z}}\xi_{\mathbf{z}}\xi_{\mathbf{z},\mathbf{j}}$. Using the first part of the proof, we can now rewrite this as

$$\sum_a \mu_a \left(\xi_{\mathbf{i},\mathbf{z}}\xi_{\mathbf{a},\mathbf{z}^+}\right)\left(\xi_{\mathbf{z}^+,\mathbf{z}}\xi_{\mathbf{z},\mathbf{j}}\right),$$

where the coefficients μ_a lie in $\mathbf{Z}[q, q^{-1}]$. Expanding the parentheses using the product algorithm, we can finally rewrite the sum as a $\mathbf{Z}[q, q^{-1}]$-linear combination of codeterminants of dominant shape, as required. ∎

3. The Quantized Straightening Formula

The next main result will be the proof of the quantum analogue of Woodcock's straightening formula for codeterminants. The aim is to express an arbitrary codeterminant as a linear combination of standard codeterminants. Proposition 2.5 shows that it is enough to be able to express an arbitrary codeterminant of dominant shape as a linear combination of standard codeterminants.

Let $\lambda \in \Lambda^+(n,r)$, $\ell = \ell(\lambda)$ and $\mathbf{i} \in I'_\lambda \backslash I_\lambda$. Note that ℓ is of parabolic form; denote the associated parabolic subgroup by W_L. We shall construct another element $\mathbf{j} \in I(n,r)$ by analysis of the tableau $T = T_{\mathbf{i}}^\lambda$, in exactly the same way as Woodcock [**9**]. (The symbols $\mathbf{i}$, $\mathbf{j}$ and λ will remain fixed throughout the next few lemmas.)

Denote by $[\lambda]$ the Young diagram of λ, considered as a subset of $\mathbf{N}^2$. By the choice of $\mathbf{i}$, T cannot be standard, and so we know there is some $(b,c) \in [\lambda]$ satisfying $T(b,c) \geq T(b+1,c)$. Let (b,c) be lexicographically minimal with respect to this property. Denote by $[\lambda]_a$ the a-th row of $[\lambda]$, and define X_a and Y_a so that $[\lambda]_a = X_a \mathbin{\dot\cup} Y_a$ $(a \in \mathbf{n})$, where

$$\begin{array}{lcllcll} X_a &=& [\lambda]_a, & Y_a &=& \emptyset, & \text{if } a < b; \\ X_b &=& \{\langle b,h\rangle \ : \ 1 \leq h < c\} & Y_b &=& [\lambda]_b \backslash X_b; & \\ X_a &=& \{x \in [\lambda]_a : T(x) \leq T(y) \text{ for all } y \in Y_{a-1}\}, & Y_a &=& [\lambda]_a \backslash X_a, & \text{if } a > b. \end{array}$$

Put $\nu_a = |X_a \mathbin{\dot\cup} Y_{a-1}|$ for $a \in \mathbf{n+1}$, and define $X_{n+1} = Y_0 = \emptyset$. Let $a_1, \ldots, a_m$ be the elements of $\{a \in \mathbf{n+1} : \nu_a \neq 0\}$, ordered so that $\alpha < \beta \Rightarrow \nu_{a_\alpha} \geq \nu_{a_\beta}$.

Note that $m \leq n$: this is clear if $X_b = \emptyset$. If $X_b \neq \emptyset$, each entry of T in X_{b+1} is at least $b+1$ by the minimality of (b,c). By induction, each of the elements of Y_a is at least $a+1$, so $Y_n = \emptyset$.

We are now ready to define $\mathbf{j} \in I$ by the property

$$T_{\mathbf{j}}^\lambda(x) = \alpha \quad \text{if } x \in X_{a_\alpha} \mathbin{\dot\cup} Y_{a_\alpha - 1}.$$

Let $\mu = \mathrm{wt}(\mathbf{j})$. By the choice of ordering on the a_α, we have $\mu \in \Lambda^+$, and in fact $\mu \succ \lambda$ since $\nu_\alpha = \lambda_\alpha$ if $\alpha < b$, while $\nu_{b+1} > \lambda_b$.

We also define $\mathbf{j}' \in I$ by the property that

$$T_{\mathbf{j}'}^\lambda(x) = \alpha \quad \text{if } x \in X_\alpha \mathbin{\dot\cup} Y_{\alpha-1}.$$

Note that we may choose the ordering on $a_1, \ldots, a_m$ above in such a way that $\mathbf{j}'^+ = \mathbf{j}$.

Example Suppose $n = 6$, $r = 18$ and

$$T_{\mathbf{i}}^{\lambda} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 & 3 \\ \cline{1-5} 3 & 3 & 3 & 5 \\ \cline{1-4} 5 & 6 \\ \cline{1-2} \end{array}.$$

Then

$$T_{\mathbf{j}}^{\lambda} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 3 & 3 & 2 & 2 & 2 \\ \cline{1-5} 2 & 2 & 2 & 4 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} \end{array}$$

and

$$T_{\mathbf{j}'}^{\lambda} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 3 & 3 & 3 \\ \cline{1-5} 3 & 3 & 3 & 4 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} \end{array}.$$

The element $\mathbf{j}$ is given by

$$(\underbrace{1,1,1,1,1,1,1}_{X_1},\underbrace{3,3}_{X_2},\underbrace{2,2,2}_{Y_2},\underbrace{2,2,2}_{X_3},\underbrace{4}_{Y_3},\underbrace{4}_{X_4},\underbrace{5}_{Y_4}),$$

where the sections of $\mathbf{j}$ corresponding to each nonempty X_a and Y_a have been marked.

Similarly, the element $\mathbf{j}'$ is given by

$$(\underbrace{1,1,1,1,1,1,1}_{X_1},\underbrace{2,2}_{X_2},\underbrace{3,3,3}_{Y_2},\underbrace{3,3,3}_{X_3},\underbrace{4}_{Y_3},\underbrace{4}_{X_4},\underbrace{5}_{Y_4}).$$

The weights λ and μ are given by

$$\lambda = (7, 5, 4, 2, 0, 0)$$

and

$$\mu = (7, 6, 2, 2, 1, 0).$$

The following lemma is very useful for reducing questions about q-codeterminants to questions about their classical analogues.

Lemma 3.1 Suppose $\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{e} \in I(n, r)$. Assume that $a_1 \leq \cdots \leq a_r$ and $b_1 \leq \cdots \leq b_r$ (which implies that $\langle \mathbf{a}, \mathbf{b} \rangle$ and $\langle \mathbf{b}, \mathbf{a} \rangle$ are H-relaxed).

(i) Write

$$\xi_{\mathbf{a},\mathbf{b}} \xi_{\mathbf{d},\mathbf{e}} = \sum_{\mathbf{a}' \sim \mathbf{a}} c_{\mathbf{a}',\mathbf{e}} \xi_{\mathbf{a}',\mathbf{e}}.$$

Then the elements $c_{\mathbf{a}',\mathbf{e}}$ lie in $\mathbf{N}[q]$.

(ii) Write

$$\xi_{\mathbf{e},\mathbf{d}}\xi_{\mathbf{b},\mathbf{a}} = \sum_{\mathbf{e}'\sim\mathbf{e}} c'_{\mathbf{e}',\mathbf{a}}\xi_{\mathbf{e}',\mathbf{a}}.$$

Then the elements $c'_{\mathbf{e}',\mathbf{a}}$ lie in $\mathbf{N}[q]$.

Proof It is clear from the definition of the multiplication in the q-Schur algebra given in [**1**, Proposition 1.2] that a product of two basis elements is a $\mathbf{Z}[q]$-combination of other basis elements. To prove both parts of the lemma, it therefore suffices to show that the coefficients occurring lie in $\mathbf{N}[v, v^{-1}]$.

We know from [**3**, Theorem 1.11] that there is an anti-automorphism of the q-Schur algebra which sends an element $\xi_{\mathbf{i},\mathbf{j}}$ to $\xi_{\mathbf{j},\mathbf{i}}$. (Details are given in §3.5.) This implies that it is enough to prove (i), because (ii) is an immediate consequence of it.

We now concentrate on (i). We may assume that $\langle\mathbf{d},\mathbf{e}\rangle$ is an H-relaxed pair. Define A to be the matrix satisfying $e_A = \xi_{\mathbf{d},\mathbf{e}}$ and define s_A and m_A as in §1.10. The parabolic subgroup of W which is the stabiliser of $\mathbf{e}$ is denoted by $W_{\mathbf{e}}$, and we write $\mathcal{D}_{\mathbf{d},\mathbf{e}}$ for the set of distinguished right coset representatives corresponding to the parabolic subgroup which is the stabiliser of the pair $\langle\mathbf{d},\mathbf{e}\rangle$.

Any relaxed basis element $u_{\mathbf{i},\mathbf{j}}$ occurring in the product $\Pi_0 := \xi_{\mathbf{a},\mathbf{b}}\xi_{\mathbf{d},\mathbf{e}}$ must have $\mathbf{j} = \mathbf{e}$ and $\mathbf{i}\sim\mathbf{a}$. Now Proposition 1.2 implies that the coefficient of $u_{\mathbf{i},\mathbf{j}}$ in Π_0 is the same as the coefficient in the product $\xi_{\mathbf{a},\mathbf{b}}x_{\mathbf{d},\mathbf{e}}$, where $x_{\mathbf{d},\mathbf{e}} \in T^r(M_n)$ is given by

$$x_{\mathbf{d},\mathbf{e}} := v^{s_A - m_A} u_{\mathbf{d},\mathbf{e}}.\left(\sum_{w\in W_{\mathbf{e}}\cap\mathcal{D}_{\mathbf{d},\mathbf{e}}} T_w\right).$$

Because we assumed that $\langle\mathbf{d},\mathbf{e}\rangle$ was H-relaxed, we find by induction on left subwords of w and iteration of equation (1) in §1.6 that if T_w is one of the elements in the sum, then

$$u_{\mathbf{d},\mathbf{e}}.T_w = v^* u_{\mathbf{d}w,\mathbf{e}w},$$

for some integer denoted by $*$. It follows that $x_{\mathbf{d},\mathbf{e}}$ is an $\mathbf{N}[v,v^{-1}]$ combination of basis elements of tensor matrix space.

Now consider $e_B = \xi_{\mathbf{a},\mathbf{b}}$ as an element of tensor matrix space. We find from Proposition 1.2 that

$$\xi_{\mathbf{a},\mathbf{b}} = v^{s_B - m_B} u_{\mathbf{a},\mathbf{b}}.\left(\sum_{w\in\mathcal{D}_{\mathbf{a},\mathbf{b}}} T_w\right).$$

We find that for each w in the sum,

$$u_{\mathbf{a},\mathbf{b}}.T_w = v^* u_{\mathbf{a}w,\mathbf{b}w}.$$

This is proved using the hypotheses on the pair $\langle \mathbf{a}, \mathbf{b} \rangle$, induction on left subwords of w, and iteration of equation (1) in §1.6. An argument similar to the one in the preceding paragraph shows that $\xi_{\mathbf{a},\mathbf{b}}$ is an $\mathbf{N}[v, v^{-1}]$-combination of basis vectors of tensor matrix space.

It is now immediate from Proposition 1.2 (iii) that the elements $c_{\mathbf{a}',\mathbf{e}}$ lie in $\mathbf{N}[v, v^{-1}]$, which completes the proof. ∎

Definitions Let Π_0 be the element of $S_q(n,r)$ given by

$$\xi_{\mathbf{i},\mathbf{j}'}\xi_{\mathbf{j}',\mathbf{l}},$$

and let Π_1 be the element of $S(n,r)$ given by the same expression.

Let Π_2 be the element of $S(n,r)$ given by

$$\xi_{\mathbf{i},\mathbf{j}}\xi_{\mathbf{j},\mathbf{l}}.$$

Lemma 3.2 Express Π_0 in terms of the basis elements of the q-Schur algebra $\{\xi_{\mathbf{a},\mathbf{b}}\}$ (where the pairs $\langle \mathbf{a}, \mathbf{b} \rangle$ are H-relaxed) by writing

$$\Pi_0 = \sum_{\mathbf{i}' \in I'_\lambda} c_{\mathbf{i}',\mathbf{l}} \xi_{\mathbf{i}',\mathbf{l}},$$

where $c_{\mathbf{i}',\mathbf{l}} \in \mathbf{Z}[q]$. Then $c_{\mathbf{i},\mathbf{l}}$ is a power of q and $\mathbf{i}' \succeq \mathbf{i}$ whenever $c_{\mathbf{i}',\mathbf{l}}$ is nonzero.

Proof Note that the product $\Pi_0 = \xi_{\mathbf{i},\mathbf{j}'}\xi_{\mathbf{j}',\mathbf{l}}$ satisfies the hypotheses of Lemma 3.1 (ii). Write $c'_{\mathbf{i}',\mathbf{l}}$ for the integer which is the specialisation as q is replaced by 1 of $c_{\mathbf{i}',\mathbf{l}}$. It is now enough in the light of Lemma 3.1 to prove that the coefficient $c'_{\mathbf{i},\mathbf{l}}$ in Π_1 is 1 and that $\mathbf{i}' \succeq \mathbf{i}$ whenever $c'_{\mathbf{i}',\mathbf{l}} \neq 0$, because $c'_{\mathbf{i},\mathbf{l}}$ is equal to 1 if and only if $c_{\mathbf{i},\mathbf{l}}$ is a (nonnegative) power of q, and $c'_{\mathbf{i}',\mathbf{l}}$ is nonzero if and only if $c_{\mathbf{i}',\mathbf{l}}$ is nonzero. (This is true because the coefficients in the statement of Lemma 3.1 all lie in $\mathbf{N}[q]$.)

It is immediate from Schur's product rule and the construction of $\mathbf{j}$ and $\mathbf{j}'$ that Π_1 and Π_2 are equal. The element Π_2 was studied in [**9**], where it was proved that the coefficient of $\xi_{\mathbf{i},\mathbf{l}}$ in Π_2 was equal to 1 and that any basis element $\xi_{\mathbf{i}',\mathbf{l}}$ appearing in Π_2 satisfied $\mathbf{i}' \succeq \mathbf{i}$.

The proof now follows. ∎

Lemma 3.3 The element Π_0 can be written as an $\mathcal{A}$-linear sum of q-codeterminants of shape μ.

Proof We see from Lemma 2.4 (i) that

$$\tau_{\mathbf{j}',\mathbf{j}}\tau_{\mathbf{j},\mathbf{j}'} = v^*\tau_{\mathbf{j}',\mathbf{j}'}$$

for some integer denoted by $*$. It follows that we can write

$$\Pi_0 = \xi_{\mathbf{i},\mathbf{j}'}\xi_{\mathbf{j}',\mathbf{l}} = \xi_{\mathbf{i},\mathbf{j}'}\xi_{\mathbf{j}',\mathbf{j}'}\xi_{\mathbf{j}',\mathbf{l}} = v^*\left(\xi_{\mathbf{i},\mathbf{j}'}\tau_{\mathbf{j}',\mathbf{j}}\right)\left(\tau_{\mathbf{j},\mathbf{j}'}\xi_{\mathbf{j}',\mathbf{l}}\right).$$

Now we expand each pair of parentheses in terms of the basis $\xi_{\mathbf{a},\mathbf{b}}$ to obtain an $\mathcal{A}$-linear sum of q-codeterminants of shape $\mu = \mathrm{wt}(\mathbf{j})$. ∎

Lemma 3.4 Let $\lambda \in \Lambda^+(n,r)$, $\ell = \ell(\lambda)$, and $\mathbf{i} \in I'_\lambda \backslash I_\lambda$. Then there exist elements $c_{\mathbf{i}',\ell} \in \mathcal{A}$ and $\mathbf{j} \in I(n,r)$ such that $\mu = \mathrm{wt}(\mathbf{j}) \in \Lambda^+$, $\mu \succ \lambda$, and

$$\xi_{\mathbf{i},\ell} = q^*\Pi_0 + \sum_{\substack{\mathbf{i}' \in I'_\lambda \\ \mathbf{i}' \succ \mathbf{i}}} c_{\mathbf{i}',\ell}\xi_{\mathbf{i}',\ell}.$$

Thus

$$\xi_{\mathbf{i},\ell} = \sum_{\substack{\mathbf{a} \sim \mathbf{i} \\ \mathbf{b} \sim \ell}} c'_{\mathbf{a},\mathbf{b}}Y^\mu_{\mathbf{a},\mathbf{b}} + \sum_{\substack{\mathbf{i}' \in I'_\lambda \\ \mathbf{i}' \succ \mathbf{i}}} c_{\mathbf{i}',\ell}\xi_{\mathbf{i}',\ell},$$

where the scalars c and c' lie in $\mathcal{A}$.

Proof The elements $\mathbf{j}$ and μ are constructed from $\mathbf{i}$ and λ as previously, where it was also shown that $\mu \in \Lambda^+$ and $\mu \succ \lambda$. The first assertion follows from rearranging the expression for Π_0 given by Lemma 3.2, and using the fact that the coefficient of $\xi_{\mathbf{i},\ell}$ is a power of q (i.e. a unit). The second assertion follows from the first by re-expressing Π_0 as an $\mathcal{A}$-linear sum of codeterminants of shape μ, as explained in Lemma 3.3. ∎

We require the following well-known property of q-Schur algebras in the proof of the main result.

Proposition 3.5 There is an anti-isomorphism $* : S_q(n,r) \to S_q(n,r)$ given by either

$$(\phi^d_{\lambda,\mu})^* = \phi^{d^{-1}}_{\mu,\lambda},$$

$$(e_A)^* = e_{A^\top},$$

$$\text{or} \quad (\xi_{\mathbf{a},\mathbf{b}})^* = \xi_{\mathbf{b},\mathbf{a}}.$$

Thus $(Y^{\nu}_{\mathbf{a},\mathbf{b}})^* = Y^{\nu}_{\mathbf{b},\mathbf{a}}$.

Proof For the proof of this with respect to the ϕ, see [**3**, Theorem 1.11]. The other two equivalent definitions follow from the equivalence of the three bases.

The last assertion is immediate from the definition of codeterminants. ∎

Definition Following [**9**], we define

$$\mathcal{B} := \{(\lambda, \mathbf{i}, \mathbf{j}) : \lambda \in \Lambda^+, \mathbf{i}, \mathbf{j} \in I'_\lambda\},$$

and we partially order $\mathcal{B}$ by $(\lambda, \mathbf{i}, \mathbf{j}) \preceq (\mu, \mathbf{i}', \mathbf{j}')$ if and only if $\lambda \prec \mu$ or ($\lambda = \mu$, $\mathbf{i} \preceq \mathbf{i}'$ and $\mathbf{j} \preceq \mathbf{j}'$).

We can now state and prove the main theorem.

Theorem 3.6 (the Straightening Formula for Quantized Codeterminants)

Let $Y^{\lambda}_{\mathbf{i},\mathbf{j}}$ be any codeterminant. Then $Y^{\lambda}_{\mathbf{i},\mathbf{j}}$ is a $\mathbf{Z}[q, q^{-1}]$-linear combination of standard codeterminants.

Proof By Proposition 2.5, it is enough to prove the theorem in the case where $\lambda \in \Lambda^+$. Since

$$S_v(n,r) \cong S_q(n,r) \oplus v.S_q(n,r)$$

as a $\mathbf{Z}[q, q^{-1}]$ module, and all codeterminants $Y^{\mu}_{\mathbf{a},\mathbf{b}}$ lie in $S_q(n,r)$, we see that it is enough to prove the result that $Y^{\lambda}_{\mathbf{i},\mathbf{j}}$ is an $\mathcal{A}$-linear combination of standard codeterminants.

Suppose $Y^{\lambda}_{\mathbf{i},\mathbf{j}}$ is not standard. It is enough to show that we can write it as an $\mathcal{A}$-linear combination of codeterminants corresponding to elements of $\mathcal{B}$ which are strictly higher in the partial order on $\mathcal{B}$, and then use induction on the poset $(\mathcal{B}, \preceq)$.

Assume first that $\mathbf{i} \notin I_\lambda$. Construct the elements $\mathbf{j} \in I(n,r)$ and $\mu \in \Lambda^+$ as in Lemma 3.4, and set $\ell = \ell(\lambda)$.

By definition, $Y^{\lambda}_{\mathbf{i},\mathbf{j}} = \xi_{\mathbf{i},\ell} \times \xi_{\ell,\mathbf{j}}$. We now substitute for $\xi_{\mathbf{i},\ell}$ using the formula in Lemma 3.4 to obtain

$$Y^{\lambda}_{\mathbf{i},\mathbf{j}} = \left(\sum_{\substack{\mathbf{a}\sim\mathbf{i} \\ \mathbf{b}\sim\ell}} c'_{\mathbf{a},\mathbf{b}} Y^{\mu}_{\mathbf{a},\mathbf{b}} \times \xi_{\ell,\mathbf{j}} \right) + \left(\sum_{\substack{\mathbf{i}' \in I'_\lambda \\ \mathbf{i}' \succ \mathbf{i}}} c_{\mathbf{i}',\ell} \xi_{\mathbf{i}',\ell} \xi_{\ell,\mathbf{j}} \right),$$

where the scalars c and c' all lie in $\mathcal{A}$. Next, we can rewrite the expression in the left-hand set of parentheses using the fact that

$$Y^{\mu}_{\mathbf{a},\mathbf{b}}\xi_{\ell,\mathbf{j}} = \xi_{\mathbf{a},\ell(\mu)}(\xi_{\ell(\mu),\mathbf{b}}\xi_{\ell,\mathbf{j}}),$$

and then expanding the product $\xi_{\ell(\mu),\mathbf{b}}\xi_{\ell,\mathbf{j}}$ as a sum of basis elements $\xi_{\ell(\mu),\mathbf{d}}$. Finally we can write the product $Y^{\mu}_{\mathbf{a},\mathbf{b}}\xi_{\ell,\mathbf{j}}$ as an $\mathcal{A}$-linear sum of codeterminants $Y^{\mu}_{\mathbf{a},\mathbf{d}}$. This gives

$$Y^{\lambda}_{\mathbf{i},\mathbf{j}} = \left(\sum_{\substack{\mathbf{a}\sim\mathbf{i}\\ \mathbf{d}\sim\mathbf{j}}} c''_{\mathbf{a},\mathbf{d}} Y^{\mu}_{\mathbf{a},\mathbf{d}}\right) + \left(\sum_{\substack{\mathbf{i}'\in I'_{\lambda}\\ \mathbf{i}'\succ\mathbf{i}}} c_{\mathbf{i}',\ell} Y^{\ell}_{\mathbf{i}',\mathbf{j}}\right),$$

where the scalars c'' and c all lie in $\mathcal{A}$.

Note that we have now succeeded in expressing $Y^{\lambda}_{\mathbf{i},\mathbf{j}}$ as the sum of codeterminants corresponding to strictly higher elements of $\mathcal{B}$, as required.

The other case, where $\mathbf{i} \in I_{\lambda}$ but $\mathbf{j} \notin I_{\lambda}$ follows by symmetry by using Proposition 3.5 followed by the preceding argument, then Proposition 3.5 again. This completes the proof.

∎

Corollary 3.7 The set of standard codeterminants forms a free $\mathbf{Z}[q, q^{-1}]$-basis of $S_q(n, r)$.

Proof The cardinality of this set is $\binom{n^2+r-1}{r}$, because it is the same as the cardinality of the corresponding set in the classical case. (See [**5**, §2] for a proof of the latter.) This is the same as the dimension of $S_q(n, r)$. It is also known (see Dipper and James [**3**, Theorem 1.4]) that the elements e_A form a free basis for $S_q(n, r)$. It is therefore enough to show that each e_A is a $\mathbf{Z}[q, q^{-1}]$-linear combination of standard codeterminants. Theorem 3.6 shows that it is enough to prove that each basis element $\xi_{\mathbf{i},\mathbf{j}}$ is a codeterminant. This is clear, because $\xi_{\mathbf{i},\mathbf{j}} = \xi_{\mathbf{i},\mathbf{j}}\xi_{\mathbf{j}}$. ∎

We end this paper by giving a brief summary of the steps needed to straighten an arbitrary codeterminant, $Y^{\lambda'}_{\mathbf{i}',\mathbf{k}'}$.

(1) Find the element $\mathbf{z} \in I(n, r)$ satisfying $a < b \Rightarrow z_a \leq z_b$ and $\mathrm{wt}(\mathbf{z}) = \lambda'$. Use the identity $\tau_{\mathbf{z},\mathbf{z}^+}\tau_{\mathbf{z}^+,\mathbf{z}} = v^*\tau_{\mathbf{z},\mathbf{z}}$ of Lemma 2.4 (i) and the techniques of Proposition 2.5 to express the codeterminant as a sum of codeterminants of dominant shape.

(2) Straighten each non-standard codeterminant $Y^\lambda_{\mathbf{i},\mathbf{k}}$ arising from the previous procedure. If $\mathbf{i} \in I_\lambda$ but $\mathbf{k} \notin I_\lambda$, straighten the codeterminant $Y^\lambda_{\mathbf{k},\mathbf{i}}$ instead and use Proposition 3.5 to "turn the results around" at the end.

(3) We may now concentrate on straightening codeterminants of form $Y^\lambda_{\mathbf{i},\mathbf{k}}$, where λ is a dominant weight and $\mathbf{i} \notin I_\lambda$. Construct the elements $\mathbf{j}$ and $\mathbf{j}'$ from $\mathbf{i}$ and λ, as described at the beginning of §3. Using the formula for $\xi_{\mathbf{i},\ell}$ given in Lemma 3.4, and the techniques of the proof of Theorem 3.6, rewrite $Y^\lambda_{\mathbf{i},\mathbf{k}}$ as a sum of codeterminants corresponding to strictly higher elements of the poset $\mathcal{B}$.

(4) Iterate parts (2) and (3) until all codeterminants appearing in the sum are standard.

ACKNOWLEDGEMENTS

The work for this paper was done while the author was supported by a grant from the SERC, and was supervised by Professor R.W. Carter at the University of Warwick.

The author thanks the referee for some useful suggestions.

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