ON RANK FUNCTIONS FOR HEAPS

R.M. Green

Department of Mathematics and Statistics
Lancaster University
Lancaster LA1 4YF
England
E-mail: r.m.green@lancaster.ac.uk

Abstract. Motivated by work of Stembridge, we study rank functions for Viennot's heaps of pieces. We produce a simple and sufficient criterion for a heap to be a ranked poset and apply the results to the heaps arising from fully commutative words in Coxeter groups.

To appear in the Journal of Combinatorial Theory, Series A

Introduction

A heap is an isomorphism class of labelled posets satisfying certain axioms. Heaps have a wide variety of applications, as discussed by Viennot in [**7**]. Stembridge [**5**] showed how to associate heaps to fully commutative elements of Coxeter groups; the latter are the elements for which any reduced expression may be obtained from any other by iterated commutation of adjacent Coxeter generators. In [**6**], Stembridge applied these ideas to λ-minuscule elements of Coxeter groups; these were first introduced by D. Peterson (unpublished) and were shown to be fully commutative by Proctor [**4**].

It follows from [**6**, Corollary 3.4] that, under the extra assumption that the labels occurring in the heap index an acyclic subset of the Coxeter graph, the heap of a minuscule element is ranked as an abstract poset. In the light of this result, it is natural to ask under what circumstances a heap is ranked, and furthermore, what can be said about the case of heaps of fully commutative elements of Coxeter groups? We maintain the assumption of [**6**, Corollary 3.4]—because, as we explain

1991 *Mathematics Subject Classification.* 06A07.

in §2.1, the situation becomes much more complicated otherwise—and we obtain in Theorem 2.1.1 a simple necessary and sufficient condition for a heap to be ranked, which involves the consideration of certain subintervals. We also look in §3 at the special case of heaps of fully commutative elements of finite Coxeter groups, where our necessary and sufficient condition can be refined so that it is more explicit and easier to verify (Theorem 3.2.3). For a Coxeter group of type A, the situation is simpler still and our main results are already known in this case (see Remark 3.3.7).

In the computer science literature [**3**], heaps have been used to model concurrency, where the elements of the heap represent processes. It would be interesting to know if rank functions for heaps have implications for the scheduling of such processes.

1. Preliminaries

1.1 Heaps.

We start by recalling the basic definitions associated to heaps. Our notation largely follows that of [**7**].

Definition 1.1.1. Let P be a set equipped with a symmetric and reflexive binary relation $\mathcal{C}$. The elements of P are called *pieces*, and the relation $\mathcal{C}$ is called the *concurrency relation.*

A *labelled heap* with pieces in P is a triple $(E, \leq, \varepsilon)$ where $(E, \leq)$ is a finite (possibly empty) partially ordered set with order relation denoted by $\leq$ and ε is a map $\varepsilon : E \longrightarrow P$ satisfying the following two axioms.

1. For every $\alpha, \beta \in E$ such that $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$, α and β are comparable in the order $\leq$.
2. The order relation $\leq$ is the transitive closure of the relation $\leq_{\mathcal{C}}$ such that for all $\alpha, \beta \in E$, $\alpha \ \leq_{\mathcal{C}}\ \beta$ if and only if both $\alpha \leq \beta$ and $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$.

The terms *minimal* and *maximal* applied to the elements of the labelled heap refer to minimality (respectively, maximality) with respect to $\leq$.

Example 1.1.2. Let $P = \{1, 2, 3\}$ and, for $x, y \in P$, define $a\ \mathcal{C}\ b$ if and only if $|x - y| \leq 1$. Let $E = \{a, b, c, d, e\}$ partially ordered by extension of the relations $a \leq c$, $b \leq c$, $c \leq d$, $c \leq e$. Define the map ε by the conditions $\varepsilon(a) = \varepsilon(d) = 1$, $\varepsilon(c) = 2$ and $\varepsilon(b) = \varepsilon(e) = 3$. Then $(E, \leq, \varepsilon)$ can easily be checked to satisfy the axioms of Definition 1.1.1 and it is a labelled heap. The minimal elements are a and b, and the maximal elements are d and e.

Definition 1.1.3. Let $(E, \leq, \varepsilon)$ and $(E', \leq', \varepsilon')$ be two labelled heaps with pieces in P and with the same concurrency relation, $\mathcal{C}$. Two labelled heaps are isomorphic if there is a poset isomorphism $\phi : E \longrightarrow E'$ such that $\varepsilon = \varepsilon' \circ \phi$ (i.e., a labelled poset isomorphism).

A *heap* of pieces in P with concurrency relation $\mathcal{C}$ is a labelled heap (Definition 1.1.1) defined up to labelled poset isomorphism. The set of such heaps is denoted by $H(P, \mathcal{C})$. We denote the heap corresponding to the labelled heap $(E, \leq, \varepsilon)$ by $[E, \leq, \varepsilon]$.

We will sometimes abuse language and speak of the underlying set of a heap, when what is meant is the underlying set of one of its representatives.

Definition 1.1.4. Let $(E, \leq, \varepsilon)$ be a labelled heap with pieces in P and F a subset of E. Let ε' be the restriction of ε to F. Let $\mathcal{R}$ be the relation defined on F by $\alpha \mathrel{\mathcal{R}} \beta$ if and only if $\alpha \leq \beta$ and $\varepsilon(\alpha) \mathrel{\mathcal{C}} \varepsilon(\beta)$. Let $\leq'$ be the transitive closure of $\mathcal{R}$. Then $(F, \leq', \varepsilon')$ is a labelled heap with pieces in P. The heap $[F, \leq', \varepsilon']$ is called a *subheap* of $[E, \leq, \varepsilon]$.

We will often implicitly use the fact that a subheap is determined by its set of vertices and the heap it comes from.

Definition 1.1.5. Let $E = [E, \leq_E, \varepsilon]$ and $F = [F, \leq_F, \varepsilon']$ be two heaps in $H(P, \mathcal{C})$. We define the heap $G = [G, \leq_G, \varepsilon''] = E \circ F$ of $H(P, \mathcal{C})$ as follows.

1. The underlying set G is the disjoint union of E and F.
2. The labelling map ε'' is the unique map $\varepsilon'' : G \longrightarrow P$ whose restriction to E (respectively, F) is ε (respectively, ε').
3. The order relation $\leq_G$ is the transitive closure of the relation $\mathcal{R}$ on G, where $\alpha \mathrel{\mathcal{R}} \beta$ if and only if one of the following three conditions holds:

(i) $\alpha, \beta \in E$ and $\alpha \leq_E \beta$;

(ii) $\alpha, \beta \in F$ and $\alpha \leq_F \beta$;

(iii) $\alpha \in E,\ \beta \in F$ and $\varepsilon(\alpha) \mathrel{\mathcal{C}} \varepsilon'(\beta)$.

Remark 1.1.6. Definition 1.1.5 can easily be shown to be sound (see [**7**, §2]). It is immediate from the construction that E and F are subheaps of $E \circ F$.

As in [**7**], we will write $\alpha \circ E$ and $E \circ \alpha$ for $\{\alpha\} \circ E$ and $E \circ \{\alpha\}$, respectively. Note that $\alpha \circ E$ and $\beta \circ E$ are equal as heaps if $\varepsilon(\alpha) = \varepsilon(\beta)$.

Definition 1.1.7. The *concurrency graph* associated to the class of heaps $H(P, \mathcal{C})$ is the graph whose vertices are the elements of P and for which there is an edge from $v \in P$ to $w \in P$ if and only if $v \neq w$ and $v \ \mathcal{C} \ w$. If $E = [E, \leq, \varepsilon]$ is a heap of $H(P, \mathcal{C})$, we define the *concurrency subgraph* of E to be the full subgraph of the concurrency graph of $H(P, \mathcal{C})$ that contains the vertices $\{\varepsilon(a) : a \in E\}$.

1.2 Rank functions.

We now give our definition of the rank function and develop some of its elementary properties.

Definition 1.2.1. Let $(E, \leq)$ be a poset. If $a, b \in E$, the relation $a < b$ is said to be a *covering relation* if there does not exist $c \in E$ such that $a < c < b$. A function $\rho : E \longrightarrow \mathbb{Z}$ is said to be a *rank function* for $(E, \leq)$ if whenever $a, b \in E$ are such that $a < b$ is a covering relation, we have $\rho(b) = \rho(a) + 1$. If a rank function for $(E, \leq)$ exists, we say $(E, \leq)$ is *ranked.*

There are variants of Definition 1.2.1 in the literature, but our formulation is convenient for our purposes.

Definition 1.2.2. Let $(E, \leq)$ be a poset and let $a, b \in E$. We write $a \sim_c b$ if $a < b$ is a covering relation, and we denote the equivalence relation on E generated by $\sim_c$ by $\sim$. We call the $\sim$-equivalence classes of E the *connected components* of $(E, \leq)$.

The following lemma is clear from the definitions.

Lemma 1.2.3. *Let $(E, \leq)$ be a poset and let $\kappa : E \longrightarrow \mathbb{Z}$ be a function constant on $\sim$-equivalence classes. If ρ is a rank function for $(E, \leq)$, then so is the function $\rho + \kappa$ defined by $(\rho + \kappa)(z) = \rho(z) + \kappa(z)$.* $\square$

Definition 1.2.4. Let $(E, \leq, \varepsilon)$ be a labelled heap. We say $(E, \leq, \varepsilon)$ is ranked if the underlying poset $(E, \leq)$ is ranked. In this case, we also say that the heap $[E, \leq, \varepsilon]$ is ranked.

Definition 1.2.4 is sound because the property of being ranked is an invariant of poset isomorphism.

Definition 1.2.5. Let $(E, \leq)$ be a poset and let $a, b \in E$. The interval $[a, b]$ is the subset $\{x \in E : a \leq x \leq b\}$. We make the same definition if $(E, \leq, \varepsilon)$ is a labelled heap. If $[E, \leq, \varepsilon]$ is the corresponding heap, we call the subheap corresponding to the subset $[a, b]$ a *subinterval* of $[E, \leq, \varepsilon]$; we will often abuse notation and refer to the subheap itself as $[a, b]$. If $[a, b]$ is a subinterval in the heap $[E, \leq, \varepsilon]$, we say $[a, b]$ is a *balanced subinterval* if $\varepsilon(a) = \varepsilon(b)$. A balanced subinterval $[a, b]$ is said to be *minimal* if $a \neq b$ and if the only elements $c \in [a, b]$ with $\varepsilon(c) = \varepsilon(a)(= \varepsilon(b))$ are $c = a$ and $c = b$.

We will regard subintervals of posets as subposets, in the obvious way. The following property will often be useful.

Remark 1.2.6. If $[a, b]$ is a subinterval in $(E, \leq)$ and $x < y$ is a covering relation in the subinterval $[a, b]$ then $x < y$ is a covering relation in $(E, \leq)$.

Lemma 1.2.7. *If $(E, \leq)$ is a ranked poset then every subinterval of $(E, \leq)$ is ranked.*

Proof. Let $a, b \in E$ with $a < b$, and let ρ be a rank function for $(E, \leq)$. Then the restriction of ρ to $[a, b]$ is a rank function for $[a, b]$ by Remark 1.2.6. $\square$

The main purpose of this paper is to investigate the extent to which the converse of Lemma 1.2.7 holds; we will see that the converse is false in general. The proof of the main result (Theorem 2.1.1) will involve the following lemma.

Lemma 1.2.8. *Let $E = [E, \leq, \varepsilon]$ be a nonempty heap in $H(P, \mathcal{C})$ and let $\alpha, \beta \in E$. If $\alpha \sim \beta$ (as in Definition 1.2.2) then there is a sequence*

$$\alpha = \gamma_0, \gamma_1, \ldots, \gamma_r = \beta$$

of elements of E such that for each $0 \leq i < r$, we have $\varepsilon(\gamma_i)\ \mathcal{C}\ \varepsilon(\gamma_{i+1})$.

Proof. Since $\alpha \sim \beta$, the definition of $\sim$ shows that there is a (possibly trivial) sequence $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_r = \beta$ where, for each $0 \leq i < r$, either $\gamma_i < \gamma_{i+1}$ or $\gamma_i > \gamma_{i+1}$ is a covering relation. The lemma now follows from part 2 of Definition 1.1.1. $\square$

2. A sufficient condition for a heap to be ranked

We devote §2 to investigating the converse of Lemma 1.2.7 for a general heap. The main result of this section is Theorem 2.1.1.

2.1. The main result.

Theorem 2.1.1. *Let $E = [E, \leq, \varepsilon]$ be a heap in $H(P, \mathcal{C})$. Suppose the concurrency subgraph of E (see Definition 1.1.7) contains no circuits. Then the following are equivalent:*

(i) *E is ranked;*

(ii) *every subinterval of E is ranked;*

(iii) *every minimal balanced subinterval of E is ranked.*

Remark 2.1.2. The implication (i) $\Rightarrow$ (ii) is immediate from Lemma 1.2.7 and the implication (ii) $\Rightarrow$ (iii) is trivial, so our strategy will be to show that (iii) implies (i).

Remark 2.1.3. The circuit avoidance property above is called property (H4) in [**6**]. Some restriction is necessary here (see Example 2.1.5), although the condition given is too strong (see Example 2.1.4).

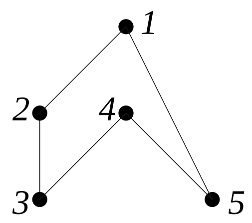
Example 2.1.4. Let $P = \{1, 2, 3, 4, 5\}$ with concurrency relation $\mathcal{C}$ such that $a \ \mathcal{C} \ b$ for all $a, b \in P$; the concurrency graph Γ is thus the complete graph on 5 vertices. Let $E = [E, \leq, \varepsilon]$ be any of the heaps of $H(P, \mathcal{C})$ with concurrency subgraph equal to Γ. In this case, $(E, \leq)$ is totally ordered, and it follows that E is a ranked heap, as are all of its subintervals. However, Γ contains circuits.

Example 2.1.5. Let $P = \{1, 2, 3, 4, 5\}$ as in Example 2.1.4, but define the concurrency relation $\mathcal{C}$ so that $a \ \mathcal{C} \ b$ if and only if $\{a, b\}$ is in the list

$$\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}.$$

In this case, Γ is a pentagon. Figure 1 shows the Hasse diagram of a heap E with concurrency subgraph Γ. (This notation is familiar from [**6**]: for example we can see from the diagram that the two minimal elements of E are labelled 3 and 5, and the two maximal elements are labelled 1 and 4.) It is not hard to see that no rank function for E exists, but that all subintervals of E are ranked. This is possible because the concurrency subgraph of E contains a circuit.

FIGURE 1. The heap E of Example 2.1.5



2.2 Proof of the main result.

Lemma 2.2.1. *Let $E = [E, \leq, \varepsilon]$ be a nonempty heap in $H(P, \mathcal{C})$, and let α be a minimal element of E. Let F be the subheap of E corresponding to the subset $E \backslash \{\alpha\}$, so that $E = \alpha \circ F$. Suppose that F is ranked and that every minimal balanced subinterval of E is ranked, and suppose further that the concurrency subgraph of E contains no circuits. If $\beta, \gamma \in F$ are in the same connected component of F and $\alpha < \beta$ and $\alpha < \gamma$ are covering relations in E, then we have $\rho(\beta) = \rho(\gamma)$ for any rank function ρ of F.*

Proof. We may assume that F is not empty and that $\beta \neq \gamma$, or there is nothing to prove. Let Γ be the concurrency subgraph of E; it contains no circuits by hypothesis. The condition $\beta \neq \gamma$ and Definition 1.1.1 imply that the pieces $\varepsilon(\beta)$, $\varepsilon(\alpha)$ and $\varepsilon(\gamma)$ are distinct; since $\alpha < \beta$ and $\alpha < \gamma$ are also covering relations, it must be the case that $(\varepsilon(\beta), \varepsilon(\alpha), \varepsilon(\gamma))$ is a sequence of distinct, adjacent vertices in Γ.

By Lemma 1.2.8, there is a sequence

$$\beta = \gamma_0, \gamma_1, \ldots, \gamma_r = \gamma$$

of elements of $F = [F, \leq, \varepsilon]$ such that for each $0 \leq i < r$, either $\varepsilon(\gamma_i) = \varepsilon(\gamma_{i+1})$ or $\varepsilon(\gamma_i)$ is adjacent to $\varepsilon(\gamma_{i+1})$ in Γ. Since Γ contains no circuits, the remarks in the first paragraph of the proof show that every path from $\varepsilon(\beta)$ to $\varepsilon(\gamma)$ passes through $\varepsilon(\alpha)$, and therefore $\varepsilon(\gamma_i) = \varepsilon(\alpha)$ for some $0 < i < r$. This means that there is an element $\alpha' \in F$ with $\varepsilon(\alpha') = \varepsilon(\alpha)$.

The subinterval $[\alpha, \alpha']$ of E is balanced, and so E contains a minimal balanced subinterval $[\alpha, \alpha'']$ for some $\alpha'' \in F$. Now α'' is comparable to both β and γ in the partial order, and condition 1 of Definition 1.1.1 implies that $\beta < \alpha''$ and $\gamma < \alpha''$. Since $\beta \in [\alpha, \alpha'']$, there must be a sequence

$$\beta = \beta_0 < \beta_1 < \cdots < \beta_t = \alpha''$$

where each of the relations $\beta_i < \beta_{i+1}$ is a covering relation in $[\alpha, \alpha'']$, and therefore (by Remark 1.2.6) in E.

Note that $[\alpha, \alpha'']$ is ranked as a subinterval of E by hypothesis; this implies that the saturated chains from α to α'' have a common length. Fixing a rank function ρ for F, we now find that $\rho(\alpha'') = \rho(\beta) + t$; similarly, $\rho(\alpha'') = \rho(\gamma) + t'$, where t' is the length of a saturated chain from γ to α''. (Note that t and t' are independent of ρ.) Because $\alpha < \beta$ and $\alpha < \gamma$ are covering relations, the above assertion about saturated chains forces $t = t'$, and we have $\rho(\beta) = \rho(\gamma)$ as required. $\square$

Proof of Theorem 2.1.1. By Remark 2.1.2, it is enough to prove the implication (iii) $\Rightarrow$ (i). Let $E = [E, \leq, \varepsilon]$ be a heap in $H(P, \mathcal{C})$. Suppose the concurrency subgraph of E contains no circuits and that every minimal balanced subinterval of E is ranked. The proof is by induction on $|E|$. If $|E|$ is 0 or 1, E will be ranked for trivial reasons and there is nothing to prove. We may therefore assume that $E = \alpha \circ F$ for some subheap F of E with $|F| = |E| - 1$, and suppose that ρ is a rank function for F. (It is clear that all subheaps of E will also have concurrency graphs with no circuits.)

If α is the only element in its connected component in E, we may extend ρ to E by defining $\rho(\alpha)$ arbitrarily. Otherwise, since α is minimal in E, we have covering relations $\alpha < \beta_i$ for some nonempty set $\{\beta_i\} \subset F$. If β_i and β_j are in the same connected component of F then Lemma 2.2.1 shows that $\rho(\beta_i) = \rho(\beta_j)$ By using Lemma 1.2.3 (if necessary) to adjust the values of the rank function on the connected components of F, we may assume that ρ is constant on the set $\{\beta_i\}$. The proof is completed by defining $\rho(\alpha) := \rho(\beta) - 1$ for (any) $\beta \in \{\beta_i\}$. $\square$

3. Heaps of fully commutative elements in Coxeter groups

In §3, we turn our attention to the special case of heaps that arise from fully commutative elements of Coxeter groups; these were studied by Stembridge in [**5**]. It turns out (Theorem 3.2.3) that if we restrict our attention to Coxeter groups having only finitely many fully commutative elements, it becomes easy to determine whether every minimal balanced subinterval of the heap is ranked. The result does not hold if we drop the finiteness hypothesis, and the proof relies on the classification of such Coxeter groups, but it is nevertheless potentially very helpful when checking examples by hand or by computer.

3.1 Heaps of fully commutative elements.

Definition 3.1.1. A *Coxeter group* is a pair (W, S) where S is a set and W is the group generated by S subject to the defining relations

$$(st)^{m(s,t)} = 1,$$

where $m(s, s) = 1$ for $s \in S$ and $2 \leq m(s, t) = m(t, s) \leq \infty$ for $s, t \in S$ and $s \neq t$. (For the purposes of this paper, we will always assume that the set S is finite.) The *Coxeter graph* of (W, S) has vertex set S. Two distinct vertices s, t in the Coxeter graph are joined by an edge labelled $m = m(s, t)$ if $m \geq 3$, but if $m = 3$ we omit the label on the edge by convention.

We take the following to be the definition of the heap of a fully commutative element; this is not the original definition but is equivalent to it by [**5**, Proposition 2.3]. In this paper, we are not concerned with the fully commutative elements of Coxeter groups themselves, but rather only with their heaps.

Definition 3.1.2. Let (W, S) be a Coxeter group. We define $\mathcal{C}$ by the condition

$$s \ \mathcal{C}\ t \Leftrightarrow m(s, t) \neq 2.$$

A heap $E = [E, \leq, \varepsilon]$ in $H(S, \mathcal{C})$ is the *heap of a fully commutative element of* W if and only if the following conditions hold.

1. There is no convex chain $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ in E such that $\varepsilon(\alpha_i) = s$ for all odd i and $\varepsilon(\alpha_i) = t$ for all even i, where $3 \leq m = m(s, t) < \infty$.
2. There is no covering relation $\alpha < \beta$ in E such that $\varepsilon(\alpha) = \varepsilon(\beta)$.

We say (W, S) is an *FC-finite Coxeter group* if the number of (heaps of) fully commutative elements is finite.

Remark 3.1.3. The fully commutative elements of W are in bijection with heaps satisfying the conditions of Definition 3.1.2; for an explanation see [**5**, §1.2].

Remark 3.1.4. The term "convex chain" in Definition 3.1.2 has its obvious meaning: a chain

$$\beta_1 < \beta_2 < \cdots < \beta_r$$

in E is said to be convex if, whenever $\gamma \in E$ is such that $\beta_i < \gamma < \beta_j$ for some $1 \leq i, j \leq r$, γ lies in the chain.

Example 3.1.5. Consider a Coxeter graph of type D_5, meaning that

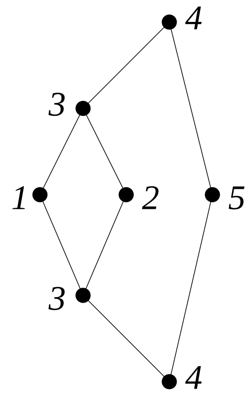
$$S = \{1, 2, 3, 4, 5\}$$

and $m(s, t) = 2$ unless $s = t$ (in which case $m(s, t) = 1$) or $\{s, t\}$ is one of the pairs

$$\{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}$$

(in which case $m(s, t) = 3$). Figure 2 shows a fully commutative heap of type D_5, i.e., of type $H(S, \mathcal{C})$ where $\mathcal{C}$ is as in Definition 3.1.2. The (unique) chain corresponding to the sequence of labels $(3, 1, 3)$ is not convex, due to the position of the occurrence of the label 2. One checks similarly that there are no chains violating condition 1 of Definition 3.1.2. It is easy to verify that the situation in condition 2 of Definition 3.1.2 cannot occur.

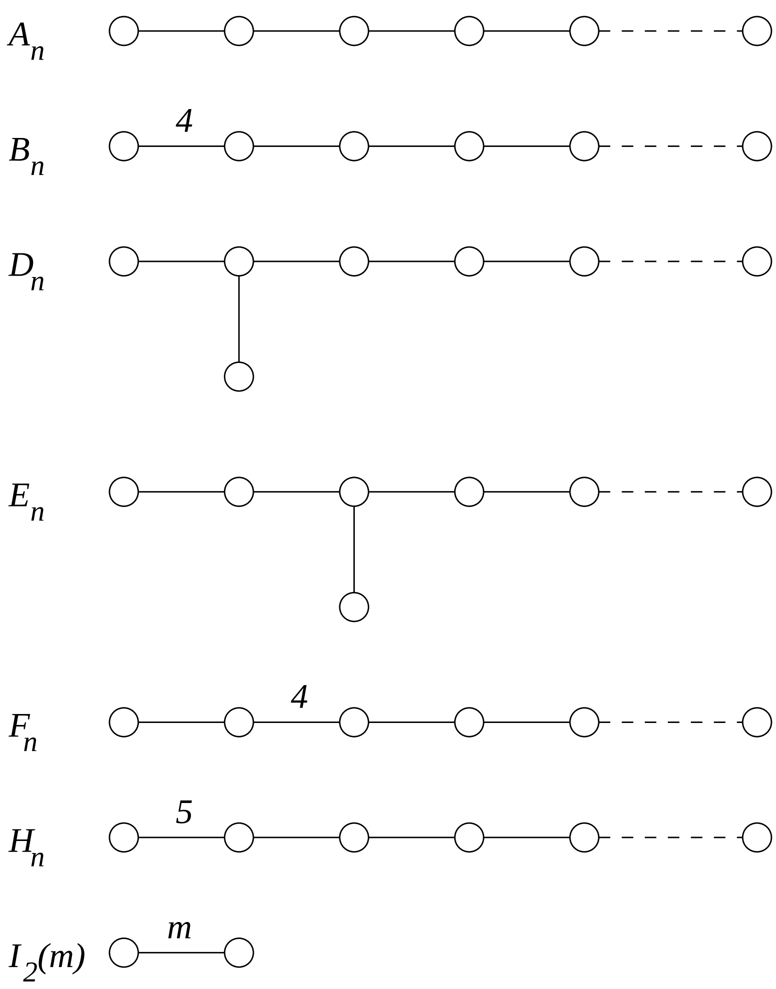
FIGURE 2. A fully commutative heap of type D_5



The classification of FC-finite Coxeter groups in terms of their Coxeter graphs was given by Stembridge [**5**, Theorem 4.1], and a similar result was independently obtained by Graham [**2**, Theorem 7.1] from an algebraic perspective.

Theorem 3.1.6 (Stembridge; Graham). *A Coxeter group (W, S) is FC-finite if and only if the connected components of its Coxeter graph appear in the list in Figure 3.* $\square$

FIGURE 3. Connected components of
Coxeter graphs of FC-finite Coxeter groups



(The subscript n in Figure 3 denotes the number of vertices in the graph.)

3.2 Ranked heaps of fully commutative elements.

The main result of §3 is Theorem 3.2.3, which gives a concise characterization of ranked heaps of fully commutative elements in FC-finite Coxeter groups.

Definition 3.2.1. Let $E = [E, \leq, \varepsilon]$ be a heap in $H(P, \mathcal{C})$, and let $[a, b]$ be a minimal balanced subinterval of E. We define the subset $S_{[a,b]}$ of E by

$$S_{[a,b]} = \{c \in [a, b] : \varepsilon(a) \neq \varepsilon(c) \text{ and } \varepsilon(a)\ \mathcal{C}\ \varepsilon(c)\}.$$

Example 3.2.2. Let a and b be the minimal and maximal elements of the heap shown in Figure 2. Then $S_{[a,b]}$ consists of three elements of the subinterval $[a, b]$ (which is in this case the whole heap): the one labelled 5 and the two labelled 3.

Theorem 3.2.3. *Let (W, S) be an FC-finite Coxeter group, and let E be the heap of a fixed fully commutative element $w \in W$. The following are equivalent:*

(i) *E is ranked;*

(ii) *for each minimal balanced subinterval $[a, b]$ of E, either (a) all the elements of $S_{[a,b]}$ have the same label or (b) all the elements of $S_{[a,b]}$ have distinct labels.*

Example 3.2.4. Let a and b be the minimal and maximal elements of the heap in Figure 2. Theorem 3.2.3 applies because a Coxeter group of type D_5 is FC-finite by Theorem 3.1.6, and the heap in question corresponds to a fully commutative element by Example 3.1.5. The three elements of $S_{[a,b]}$ do not all have the same label, but they do not have distinct labels either, so the heap is not ranked.

3.3 Proof of Theorem 3.2.3.

Definition 3.3.1. Let Γ be a Coxeter graph and let s and t be adjacent vertices of Γ. Let $\Gamma \backslash \{s\}$ be the graph obtained from Γ by deleting s and all edges emerging from s, let $\Gamma_{s,t}$ be the connected component of $\Gamma \backslash \{s\}$ that contains t, and let $\Gamma_{s \to t}$ be the full subgraph of Γ containing s and the vertices of $\Gamma_{s,t}$.

Example 3.3.2. Let Γ be a graph of type E_8 as shown in Figure 3, let s be the vertex of degree 3 and let t be the vertex immediately to the right of s. Then $\Gamma \backslash \{s\}$ consists of the disjoint union of three Coxeter graphs of types A_1, A_2 and A_4; $\Gamma_{s,t}$ is a Coxeter graph of type A_4 and $\Gamma_{s \to t}$ is a Coxeter graph of type A_5 containing s and all the vertices to the right of s.

Lemma 3.3.3. *Let Γ be the Coxeter graph of an FC-finite Coxeter group and let s be a vertex of Γ with degree strictly greater than 1. There is at most one vertex t adjacent to s such that $\Gamma_{s \to t}$ is not of type A_n for some $n \geq 2$.*

Proof. This is a case by case check using Theorem 3.1.6 (see Figure 3). □

Lemma 3.3.4. *Let $E = [E, \leq, \varepsilon]$ be the heap (in $H(P, \mathcal{C})$) of a fully commutative element in an FC-finite Coxeter group and let $[a, b]$ be a minimal balanced subinterval of E. Suppose the elements of $S_{[a,b]}$ do not all have the same label. Then there exists an element of $S_{[a,b]}$ whose label is unique among the labels of elements of $S_{[a,b]}$.*

Proof. Since the elements of $S_{[a,b]}$ do not all have the same label, the degree of $\varepsilon(a)$ in the concurrency graph Γ is greater than 1. Let $c, d \in S_{[a,b]}$ be such that $\varepsilon(c) \neq$

$\varepsilon(d)$; both labels are distinct from $\varepsilon(a) = \varepsilon(b)$ by minimality of the subinterval. By Lemma 3.3.3, we may assume without loss of generality that $\Gamma_{\varepsilon(a)\to\varepsilon(c)}$ is of type A_n for some $n \geq 2$. We index the vertices of this subgraph of type A by $p_1 = \varepsilon(a)$, $p_2 = \varepsilon(c)$, $p_3, \ldots, p_n$ such that p_i and p_j are adjacent in Γ if and only if $|i - j| = 1$.

Suppose, for a contradiction, that E is the heap of a fully commutative element in an FC-finite Coxeter group, and that $[a, b]$ is a minimal balanced subinterval of E for which (a) the elements of $S_{[a,b]}$ do not all have the same label and (b) there is no element of $S_{[a,b]}$ whose label is unique among the labels of elements in $S_{[a,b]}$. We claim by induction that for each $1 \leq k < n$, there is a minimal balanced subinterval $[a_k, b_k]$ with $\varepsilon(a_k) = \varepsilon(b_k) = p_k$ containing at least two elements labelled p_{k+1}. Define $a = a_1$, $b = b_1$ and observe that $[a, b]$ contains at least one element labelled p_2 by definition of c. By part (b) of the assertion above, there must be at least two elements of $[a, b]$ labelled p_2, which establishes the $k = 1$ case of the induction.

For the inductive step, we may assume $n > 2$. Suppose $k < n-1$ and that $[a_k, b_k]$ contains at least two elements, a' and b', labelled p_{k+1}. We may assume that the balanced chain $[a', b']$ is minimal by choosing a' and b' suitably. By minimality of $[a_k, b_k]$, we see that $[a', b']$ contains no elements labelled p_k. Since $\Gamma_{\varepsilon(a)\to\varepsilon(c)}$ is of type A_n, we must have at least two elements in $[a', b']$ labelled p_{k+2}: if there were none, we would have a counterexample to condition 2 of Definition 3.1.2 by taking $\alpha = a', \beta = b'$, and if there were only one, we would have a counterexample to condition 1 of that definition by taking $\alpha_1 = a', \alpha_3 = b'$ and α_2 to be the element labelled p_{k+2}. This proves the inductive step after taking $a_{k+1} = a'$, $b_{k+1} = b'$.

This situation leads to a contradiction because $[a_n, b_n]$ is a minimal balanced subinterval containing no occurrences of p_{n-1} (using the case $k = n - 1$ above). Taking $\alpha = a_n, \beta = b_n$ in condition 2 of Definition 3.1.2 shows that E is not the heap of a fully commutative element, a contradiction. □

Lemma 3.3.5. *Let $E = [E, \leq, \varepsilon]$ be a heap in $H(P, \mathcal{C})$ such that the concurrency subgraph of E contains no circuits, and let $[a, b]$ be a minimal balanced subinterval of E. Suppose $c \in S_{[a,b]}$ and define a' (respectively, b') to be the minimal (respectively, maximal) element of $S_{[a,b]}$ with label $\varepsilon(c)$. Then $a < a'$ and $b' < b$ are covering relations in E.*

Proof. We deal with the case of a'; the other case is similar. Since $\varepsilon(a)\ \mathcal{C}\ \varepsilon(a')$, there is a chain of covering relations

$$a = a_0 < a_1 < \cdots < a_t = a'.$$

The definition of a' ensures that $t > 0$, and we are done if $t = 1$, so suppose $t > 1$. Since $a' < b$, minimality of $[a, b]$ shows that if $i > 0$ then a_i cannot have label $\varepsilon(a)$. Similarly, the definition of a' shows that if $i < t$ then a_i cannot have label $\varepsilon(a')$. By Lemma 1.2.8, the corresponding sequence

$$\varepsilon(a_0), \varepsilon(a_1), \ldots, \varepsilon(a_t)$$

in P is a path (possibly with repeated vertices) between the adjacent vertices $\varepsilon(a_0)$ and $\varepsilon(a_t)$ that passes through each of $\varepsilon(a_0)$ and $\varepsilon(a_t)$ precisely once, which is impossible as $t > 1$ and the concurrency graph contains no circuits. This completes the proof. □

Example 3.3.6. Maintain the set-up in Example 3.2.2; recall that this concerns the heap of a fully commutative element. As noted in Example 3.2.2, the elements of $S_{[a,b]}$ do not all have the same label; Lemma 3.3.4 then predicts that one of the labels (5 in this case) occurs uniquely in the subinterval $[a, b]$. (This is because $\Gamma_{4\to5}$ is of type A_2.) Lemma 3.3.5 predicts that each of the elements labelled 3 or 5 covers or is covered by either a or b.

Proof of Theorem 3.2.3. Since (W, S) is an FC-finite Coxeter group, the concurrency graph of E has no circuits because none of the graphs in Figure 3 has any circuits. (The relation between the Coxeter graph and the concurrency graph is given in Definition 3.1.2.)

First, suppose E is ranked. By Theorem 2.1.1, every minimal balanced subinterval of E is ranked; let $[a, b]$ be such an subinterval. If all the elements of $S_{[a,b]}$ have the same label then condition (ii) of Theorem 3.2.3 holds, and we are done. If not, Lemma 3.3.4 shows the existence of an element $c \in [a, b]$ whose label is unique among the labels of $S_{[a,b]}$. By Lemma 3.3.5, $a < c$ and $c < b$ are covering relations, which means that if ρ is any rank function for E then $\rho(b) = \rho(a) + 2$. Suppose the statement of Theorem 3.2.3 (ii) does not hold, so that there exist at least two elements $d, d' \in S_{[a,b]}$ with $\varepsilon(d) = \varepsilon(d') \neq \varepsilon(c)$. Without loss of generality, $d < d'$, so we have a chain $a < d < d' < b$. This means that $\rho(b) > \rho(a)+2$, a contradiction, and condition (ii) of Theorem 3.2.3 holds, as required.

For the converse, we will prove by induction on $|E|$ that (ii) implies (i). If $|E|$ is 0 or 1 the heap E is ranked for trivial reasons and there is nothing to prove. For the general case, assume the hypotheses of (ii) and consider an arbitrary minimal balanced subinterval $[a,b]$ in E. If we can prove that $[a,b]$ is ranked, the claim will follow by Theorem 2.1.1. There are two cases to consider.

In the first case, the labels of the elements $c_1, c_2, \dots, c_r$ of $S_{[a,b]}$ are distinct. Lemma 3.3.5 shows that $a < c_i < b$ is a chain of covering relations for each i, so the subinterval $[a,b]$ consists only of the elements c_i together with a and b. The subinterval is ranked in this case: we may take $\rho(a) = 0$, $\rho(b) = 2$ and $\rho(c_i) = 1$ for each i.

In the second case to be considered, the elements $c_1, c_2, \dots, c_r$ of $S_{[a,b]}$ all have the same label, so we may assume that $c_1 < c_2 < \cdots < c_r$. By Lemma 3.3.5, $a < c_1$ and $c_r < b$ are covering relations in E; there are no other covering relations of the form $a < c'$ or $c' < b$ by the assumption on $S_{[a,b]}$. It follows that the subinterval $[a,b]$ consists (as a set) of the balanced subinterval $[c_1, c_r]$ together with the additional elements a and b. We claim that any subinterval in the heap of a fully commutative element is also the heap of a fully commutative element for the same Coxeter group: this follows from Definition 3.1.2 and the general fact that any convex chain in a subinterval of a poset is also a convex chain in the poset. Furthermore, we claim that any minimal balanced subinterval $[d,e]$ of an subinterval in a heap E is also a minimal balanced subinterval of E: it is minimal because the set of elements in E with a given label is totally ordered. These two observations show that $[c_1, c_r]$ is the heap of a fully commutative element $w \in W$, and that it satisfies condition (ii) of Theorem 3.2.3. The subinterval $[c_1, c_r]$ contains strictly fewer elements than E and is therefore ranked by the inductive hypothesis; let ρ be a rank function for $[c_1, c_r]$. We can extend ρ to a rank function for $[a,b]$ by defining $\rho(a) = \rho(c_1) - 1$ and $\rho(b) = \rho(c_r) + 1$. $\square$

Remark 3.3.7. If E is a heap of fully commutative element of a Coxeter group of type A_n, it is well known and easy to show using the techniques of the proof of Lemma 3.3.4 that if $[a,b]$ is a minimal balanced subinterval of E then $S_{[a,b]}$ consists of precisely two elements, with distinct labels. It follows that any heap of a fully commutative element of a Coxeter group of type A_n is ranked. This is also well known and is what allows Billey and Warrington's method of "pushing together the connected components of a heap" [**1**, §3] to work.

Acknowledgements

The author thanks J.R. Stembridge for helpful correspondence, and the referee for suggesting many improvements to an earlier version of this paper.

References

[**1**] S.C. Billey and G.S. Warrington, *Kazhdan–Lusztig Polynomials for 321-hexagon-avoiding permutations*, J. Algebraic Combin. **13** (2001), 111–136.

[**2**] J.J. Graham, *Modular representations of Hecke algebras and related algebras*, Ph.D. thesis, University of Sydney, 1995.

[**3**] A. Mazurkiewicz, *Trace theory*, Petri nets, applications and relationship to other models of concurrency, Lecture Notes in Computer Science 255, Springer, Berlin–Heidelberg–New York, 1987, pp. 279–324.

[**4**] R.A. Proctor, *Minuscule elements of Weyl groups, the numbers game, and d-complete posets*, J. Algebra **213** (1999), 272–303.

[**5**] J.R. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), 353–385.

[**6**] J.R. Stembridge, *Minuscule elements of Weyl groups*, J. Algebra **235** (2001), 722–743.

[**7**] G.X. Viennot, *Heaps of pieces, I: basic definitions and combinatorial lemmas*, Combinatoire Énumérative, ed. G. Labelle and P. Leroux, Springer-Verlag, Berlin, 1986, pp. 321–350.