A Matrix Setting for the q-Schur Algebra

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1. *Introduction*

The Schur algebra $S(n,r)$ has a basis (described in [**6**, §2.3]) consisting of certain elements $\xi_{i,j}$ where $i, j \in I(n,r)$, the set of all ordered r-tuples of elements from the set $\mathbf{n} = \{1, 2, \ldots, n\}$. The multiplication of two such basis elements is given by a formula known as Schur's product rule. In recent years, a q-analogue $S_q(n,r)$ of the Schur algebra has been investigated by a number of authors, particularly Dipper and James ([**3**], [**4**]). The main result of the present paper, Theorem 3.6, shows how to embed the q-Schur algebra in the r-th tensor power $T^r(M_n)$ of the $n \times n$ matrix ring. This embedding allows products in the q-Schur algebra to be computed in a straightforward manner, and gives a method for generalising results on $S(n,r)$ to $S_q(n,r)$. In particular we shall make use of this embedding in subsequent work to prove a straightening formula in $S_q(n,r)$ which generalises the straightening formula for codeterminants due to Woodcock [**12**].

We will be working mainly with three types of algebra: the quantized enveloping algebra $U(gl_n)$ corresponding to the Lie algebra gl_n, the q-Schur algebra $S_q(n,r)$, and the Hecke algebra, $\mathcal{H}(A_{r-1})$. It is often convenient, in the case of the q-Schur algebra and the Hecke algebra, to introduce a square root of the usual parameter q which shall be denoted by v, as in [**5**]. This corresponds to the parameter v in $U(gl_n)$. We will denote this "extended" version of the q-Schur algebra by $S_v(n,r)$, and we shall usually refer to it as the v-Schur algebra. All three algebras are associative and have multiplicative identities, and the base field will be the field of rational functions, $\mathbf{Q}(v)$, unless otherwise stated. The symbols n and r shall be reserved for the integers given in the definitions of these three algebras.

1.1 The Quantized Enveloping Algebra $U(gl_n)$

The algebra $U(gl_n)$ was defined, for example, in [**5**] as a $\mathbf{Q}(v)$-algebra with identity given by algebra generators

$$E_i, F_i, K_j, K_j^{-1}$$

(where $1 \leq i \leq n-1$ and $1 \leq j \leq n$) subject to certain relations.

The algebra is also equipped with a coassociative comultiplication, $\Delta : U \to U \otimes U$, which is an algebra homomorphism and has the following effect on the generators:

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i K_{i+1}^{-1},$$
$$\Delta(F_i) = K_i^{-1} K_{i+1} \otimes F_i + F_i \otimes 1,$$
$$\Delta(K_i) = K_i \otimes K_i,$$
$$\Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1}.$$

1.2 The Classical Schur Algebra

Denote by Θ_r the set of $n \times n$ matrices with nonnegative integer coefficients whose entries sum to r.

Let $I(n, r)$ be the set of all ordered r-tuples of elements from the set $\mathbf{n} := \{1, \dots, n\}$.

The classical Schur algebra, $S(n, r)$ has a basis (described in [**6**, §2.3]) consisting of certain elements $\xi_{i,j}$, where $i, j \in I(n, r)$. In this paper, we will work with another form of the this basis for $S(n, r)$, this time indexed Θ_r. A typical basis element will be denoted by e_A, where $A \in \Theta_r$.

Let $i = (i_1, \dots, i_r)$ and $j = (j_1, \dots, j_r)$ be elements of I. The identification between the two forms of the basis is given by

$$\mu(\xi_{i,j}) = e_A,$$

where the (x, y)-entry of the matrix A is given by the number of pairs (i_s, j_s) such that $i_s = x$ and $j_s = y$. It is easily seen that this is well-defined and that $A \in \Theta_r$. It is also easy to see that the map μ is surjective, and hence bijective because both bases contain the same number of elements.

1.3 The q-Schur algebra, $S_q(n, r)$

The q-Schur algebra was defined in [**5**] and [**1**, Proposition 1.2] as a $\mathbf{Q}(v)$-vector space with basis $\{e_A : A \in \Theta_r\}$, equipped with a certain associative multiplication.

We will also be using certain elements $[A]$ in $S_v(n, r)$. These are closely related to the basis elements e_A via $[A] := v^{-s} e_A$. Here, s is given by $\sum_{i,j,k,l} A_{ij} A_{kl}$, where the indices are required to satisfy $i \geq k$ and $j < l$.

1.4 The map $\theta : U(gl_n) \to S_v(n,r)$

Beilinson et al. [**1**, §5.7] define a surjective algebra homomorphism $\theta = \theta_r : U(gl_n) \to S_v(n,r)$. This makes $S_v(n,r)$ into a U-module, and shows that it is a quotient of U. This is given, following [**5**], as follows:

$$\begin{aligned}\theta(E_i) &= \sum_{D \in \mathbf{D}, E_{i,i+1}+D \in \Theta_r} [E_{i,i+1} + D],\\ \theta(F_i) &= \sum_{D \in \mathbf{D}, E_{i+1,i}+D \in \Theta_r} [E_{i+1,i} + D],\\ \theta(K_i) &= \sum_{D \in \mathbf{D}_r} v^{d_i}[D],\\ \theta(K_i^{-1}) &= \sum_{D \in \mathbf{D}_r} v^{-d_i}[D].\end{aligned}$$

Here, $\mathbf{D}$ is the set of diagonal matrices, and $\mathbf{D}_r$ means $\mathbf{D} \cap \Theta_r$. The matrix $E_{a,b}$ has 1 in the (a,b) position and zeros elsewhere.

1.5 Tensor Space

Let V be the $\mathbf{Q}(v)$-vector space with basis $(e_1, \ldots, e_n)$. Then $T^r(V) = V^{\otimes r}$ can be made into a U–$\mathcal{H}$ bimodule as follows. Firstly we can make V into a left U-module via

$$\begin{aligned}E_i e_{i+1} &= e_i,\\ E_i e_j &= 0 \quad (j \neq i+1),\\ F_i e_i &= e_{i+1},\\ F_i e_j &= 0 \quad (j \neq i),\\ K_i e_i &= v e_i,\\ K_i e_j &= e_j \quad (j \neq i).\end{aligned}$$

We now make $T^r(V)$ into a left U-module via the homomorphism $\Delta^{(r-1)}$. Following [**5**], we now make $T^r(V)$ into a right $\mathcal{H}$-module via

$$e_{i_1} \otimes \cdots \otimes e_{i_r} T_{p,p+1} := \begin{cases} v e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r} & \text{if } i_p < i_{p+1};\\ v^2 e_{i_1} \otimes \cdots \otimes e_{i_r} & \text{if } i_p = i_{p+1};\\ v e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r} + (v^2-1) e_{i_1} \otimes \cdots \otimes e_{i_r} & \text{if } i_p > i_{p+1}.\end{cases}$$

Jimbo [**9**] proved that there exists a quotient $\rho_r(U)$ of U such that $\rho_r(U) = \mathrm{End}_{\mathcal{H}}(T^r(V))$ and $\mathcal{H} = \mathrm{End}_U(V^{\otimes r})$, if $n \geq r$. It is shown in [**4**, Theorem 6.6] that the q-Schur algebra $S_q(n,r)$ is the centralising algebra of $\mathcal{H}$ with respect to "q-tensor space". It is shown in [**5**] that Dipper and James' so-called q-tensor space is essentially the same as the classical tensor space defined above, but with certain powers of v introduced. Since the rings of $\mathcal{H}$-endomorphisms of the two spaces are isomorphic, we may work with either, and in this paper, we will mainly be using classical tensor space. Thus, the v-Schur algebra can be identified with the quotient of U which acts faithfully on $T^r(V)$.

1.6 Tensor Matrix Space

Note that $M_n = M_n(R)$, the $n \times n$ matrix algebra over R (where $R = \mathbf{Q}$ for the classical case and $R = \mathbf{Q}(v)$ for the quantized case), has a natural U-module structure as follows. Denote by e_{ij} or $e_{i,j}$ the matrix unit with 1 in the (i,j)-position and zeros elsewhere. A quick check shows that there is an algebra homomorphism θ_1 from U to M_n satisfying:

$$\begin{aligned}
\theta_1(E_i) &= e_{i,i+1}, \\
\theta_1(F_i) &= e_{i+1,i}, \\
\theta_1(K_i) &= \sum_{j=1}^{n} v^{\delta(i,j)} e_{j,j}, \\
\theta_1(K_i^{-1}) &= \sum_{j=1}^{n} v^{-\delta(i,j)} e_{j,j}.
\end{aligned}$$

Note that $\theta_1 = \theta_r$ for $r = 1$ where θ_r is defined in §1.4. We will regard M_n as a left U-module via the action

$$u.m := \theta_1(u) \times m.$$

We define *tensor matrix space* to be the tensor power $T^r(M_n)$ of this matrix ring. The Coxeter group of type A_{r-1} (i.e. the symmetric group on r letters, $\mathcal{S}_r$) acts on the right of $T^r(M_n)$ by permutation of the tensors via

$$e_{i_1,j_1} \otimes \cdots \otimes e_{i_r,j_r}.\sigma := e_{i_{1.\sigma},j_{1.\sigma}} \otimes \cdots \otimes e_{i_{r.\sigma},j_{r.\sigma}}.$$

The space $T^r(M_n)$ has an associative multiplication defined on it given by

$$e_{a_1b_1} \otimes \cdots \otimes e_{a_rb_r} \times e_{c_1d_1} \otimes \cdots \otimes e_{c_rd_r} := \begin{cases} e_{a_1d_1} \otimes \cdots \otimes e_{a_rd_r} & \text{if } b_i = c_i \text{ for all } i; \\ 0 & \text{otherwise.} \end{cases}$$

We regard $T^r(M_n)$ as a U-module via the comultiplication Δ.

1.7 Hebrew Lexicographic Order

Let A and B be ordered sets. Then the *Hebrew lexicographic order* on $A \times B$ is given by $(a,b) < (c,d)$ if and only if $b < d$ or ($b = d$ and $a < c$). The elements of a sequence $(i_1, \ldots, i_m)$ in $A \times B$ are said to appear in Hebrew lexicographic order if $i_1 \leq \cdots \leq i_r$ with respect to this order.

2. The classical Schur algebra as a subspace of tensor matrix space

We now embed the Schur algebra in $T^r(M_n)$.

With each matrix A, we can (uniquely) associate a standard basis element t_A of $T^r(M_n)$, given by

$$e_{a_1b_1} \otimes \cdots \otimes e_{a_rb_r},$$

where the ordered pairs (a_i, b_i) appear in Hebrew lexicographic order, and the multiplicity with which e_{ab} occurs in t_A is the entry in the (a,b)-position of A. The symmetric group $\mathcal{S}_r$ acts on the right of the set of basis elements of $T^r(M_n)$ by place permutation of the tensors. Consider the orbit containing the basis element t_A, and let $\mathcal{D}_A$ be the set of distinguished right coset representatives relative to the stabilizer of t_A (which is a parabolic subgroup of $W(A_{r-1})$).

Example Let A be the matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then t_A is given by

$$e_{11} \otimes e_{11} \otimes e_{11} \otimes e_{21} \otimes e_{12} \otimes e_{22} \otimes e_{22}.$$

Definition We now define a map $\psi : S(n,r) \to T^r(M_n)$, given by

$$\psi(e_A) := t_A . \left(\sum_{w \in \mathcal{D}_A} w \right),$$

and extended by linearity.

It is important to notice that if A ranges over all elements of Θ_r, then $\psi(e_A)$ ranges over a basis for the subspace of $T^r(M_n)$ consisting of all elements t such that $t.\pi = t$ for all $\pi \in \mathcal{S}_r$. For example, if $n = 2$ and $r = 3$, and A is the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix},$$

then

$$\psi(e_A) = e_{11} \otimes e_{11} \otimes e_{12} + e_{11} \otimes e_{12} \otimes e_{11} + e_{12} \otimes e_{11} \otimes e_{11}.$$

From this, we see that dim (im ψ) is equal to the number of monomials of degree r in n^2 commuting variables, i.e. $\binom{n^2+r-1}{r}$.

The following results establish that ψ is a monomorphism of algebras, and of $\mathcal{U}$-modules, where $\mathcal{U}$ is the universal enveloping algebra of gl_n.

Proposition 2.1 The map $\psi : S(n,r) \to T^r(M_n)$ is a monomorphism of $\mathcal{U}$-modules.

Proof The preceding discussion of the properties of ψ shows that ψ is injective. It is known that the dimension of $S(n,r)$ is given by

$$\binom{n^2 + r - 1}{r},$$

which is the same as the number of monomials of degree r in n^2 commuting variables, and hence is equal to dim (im ψ). It suffices to check that the effect of E_i, F_i and H_i' is as expected, because these generate $\mathcal{U}$ as an algebra. (Here, H_i' denotes the specialisation of

$$\frac{K_i - K_i^{-1}}{v - v^{-1}}$$

in quantum gl_n. The element $H_i \in gl_n$ corresponding to the element $H_i \in sl_n$ is related to it via $H_i = H_i' - H_{i+1}'$.) We tackle first the case of E_h. From the definition of $\theta(E_h)$ and the specialisation of [**1**, Lemma 3.4 (a2)], we see that

$$E_h.e_A = \sum_{p:a_{h+1,p} \geq 1} (a_{h,p} + 1)\ e_{(A+e_{h,p}-e_{h+1,p})}.$$

Now consider the case of $E_h.\psi(e_A)$. Considering the number of different terms in $\psi(e_A)$ that give rise, under the action of E_h, to a particular term of $\psi(e_B)$ (for various matrices B of the correct type), we find that the same formula as above still holds.

The situation for F_h is entirely analogous, but uses [**1**, Lemma 3.4 (b2)].

It follows from the definition of $\theta(K_i)$ that

$$\frac{K_h - K_h^{-1}}{v - v^{-1}}.e_A = \left[\sum_{j=1}^{n} A_{hj}\right] e_A,$$

thus

$$H_h'.e_A = \sum_{j=1}^{n} A_{hj}\ e_A.$$

It is not hard to see that H_h' acts on $\psi(e_A)$ as exactly the same scalar multiple.

These observations suffice to prove the Proposition. ∎

Before showing that ψ is an isomorphism of algebras, we remark that our "tensor matrix space" is canonically isomorphic to an algebra with basis elements $u_{i,j}$ (where $i, j \in I(n, r)$) which appears in the literature (see [**7**, §2]). The multiplication rule for the elements $u_{i,j}$ is as follows:

$$u_{i,j} u_{k,l} = \delta_{j,k} u_{i,l},$$

where δ is the Kronecker delta. The symmetric group on r letters acts on $u_{i,j}$ via $u_{i,j}^{\pi} = u_{i\pi,j\pi}$.

There is a linear map from the algebra generated by the $u_{i,j}$ into $T^r(M_n)$ determined by

$$\sigma(u_{i,j}) = e_{i_1 j_1} \otimes \cdots \otimes e_{i_r j_r}.$$

It is immediate that σ is an algebra isomorphism.

Proposition 2.2 $\psi : S(n,r) \to T^r(M_n)$ is an algebra monomorphism.

Proof If V is a vector space of dimension n, then we have the canonical algebra isomorphism

$$c : \mathrm{End}(V^{\otimes r}) \cong (\mathrm{End}(V))^{\otimes r} (\equiv T^r(M_n))$$

sending the matrix unit $e_{i,j}$ to $e_{i_1 j_1} \otimes \cdots \otimes e_{i_r j_r}$ for all $i = (i_1, \ldots, i_r), j = (j_1, \ldots, j_r) \in I = I(n,r)$. For $A \in \Theta_r$, let $\mathcal{O}_A$ be the orbit in $I \times I$ with respect to the $\mathcal{S}_r$-action $(i,j)\pi = (i\pi, j\pi)$ corresponding to A. Then $e_A \in S(n,r)$ is the linear transformation sending e_j to e_i if $(i,j) \in \mathcal{O}_A$, and to zero otherwise (see [**6**, 2.6]). Therefore, we have clearly $c(e_A) = \psi(e_A)$ by the definition of ψ. The proof follows. ∎

The maps Δ and θ have classical counterparts obtained by specialising v and K_i (for all i) to be 1. Since no confusion is likely to arise, we use the symbols Δ and θ for both classical and quantized maps.

The preceding results can also be viewed in the context of the map $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$. We have seen that $S(n,1)$ is isomorphic as a ring and as a $\mathcal{U}$-module to M_n, and the same holds for the quantized version $S_v(n,1)$. Denote by θ_1 the map from $\mathcal{U}$ to M_n induced by $\theta : \mathcal{U} \to S(n,1)$. This leads to the following result.

Proposition 2.3 The map $\psi\theta$ agrees with the map

$$\gamma_r := \underbrace{(\theta_1 \otimes \cdots \otimes \theta_1)}_{r \text{ times}} \Delta^{(r-1)}$$

on $\mathcal{U}$.

Proof The standard $\mathcal{U}$-module V in §1.5 induces

(a) a $\mathcal{U}$-module $V^{\otimes r}$ via the comultiplication Δ and

(b) an algebra homomorphism $\theta_1 : \mathcal{U} \to \mathrm{End}(V)$ (as in §1.6).

From (a), we have an algebra homomorphism $\theta : \mathcal{U} \to \mathrm{End}(V^{\otimes r})$ whose image is the Schur algebra $S(n,r)$. From (b) we have an algebra homomorphism

$$\theta' = \theta_1^r \circ \Delta^{(r-1)} : \mathcal{U} \to T^r(M_n).$$

Clearly, we have $c \circ \theta = \theta'$, where c is the map defined in the proof of Proposition 2.2. This completes the proof. ∎

Remark We shall see that the above result quantizes, in the sense that something similar works when Δ and θ as above are replaced by their quantum analogues. This yields a quantized version of γ_r which, again, is an algebra homomorphism. Because tensor space is made into a U-module via a process equivalent to applying γ_r, and because the v-Schur algebra can be described as the faithful quotient of U corresponding to this action, it can be seen that

$$\gamma_r(U) \cong S_v(n,r),$$

as an algebra and as a U-module. This isomorphism is not as obvious as in the classical case and it is harder to work out the images of basis elements under γ_r, but, we shall see in §3 that it can be done, by finding a suitable quantum analogue of ψ.

3. The v-Schur algebra as a subspace of tensor matrix space

We have seen in the above sections that the classical Schur algebra $S(n,r)$ embeds in $T^r(M_n)$ in a natural way compatible with the $\mathcal{U}$-module and algebra structures of both objects. Moreover, the portion of $T^r(M_n)$ identified with the Schur algebra is the fixed point subspace under the action of the symmetric group $\mathcal{S}_r$ acting on the right of $T^r(M_n)$ by place permutation. Next, we define an action of the Hecke algebra $\mathcal{H}(\mathcal{S}_r)$ (corresponding to the symmetric group on r letters) on $T^r(M_n)$ on the right. The fixed point subspace of $T^r(M_n)$ corresponding to the "ind" representation of the Hecke algebra (which corresponds to the trivial representation of the symmetric group) will remarkably be seen to have a U-module structure and algebra structure which agree precisely with those of the v-Schur algebra. By specialising the parameters to 1, we recover the classical objects of the previous part of this paper.

Definition Let

$$e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r}$$

be a standard basis element of $T^r(M_n)$. We define the corresponding quantized basis element to be

$$[e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r}] := v^{-m} e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r},$$

where

$$m := |\{(i,j) : 1 \leq i < j \leq r,\ a_i > a_j\}| + |\{(i,j) : 1 \leq i < j \leq r,\ b_i > b_j\}|.$$

If $u_{i,j}$ corresponds to $e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r}$ in the way described in §2, then we write $[u_{i,j}]$ for the quantity $v^* u_{i,j}$ corresponding to $[e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r}]$. (We use the convention that the symbol $*$ represents some integer.)

Definition The action of $\mathcal{H}(A_1)$ on $T^2(M_n)$ is defined as follows. Let $[e_{ab} \otimes e_{cd}]$ be a typical (quantized) basis element. Then $[e_{ab} \otimes e_{cd}].T_s$ (s being the standard generator of the Coxeter group of type A_1) is given by:

$$[e_{ab} \otimes e_{cd}].T_s := \begin{cases} q[e_{cd} \otimes e_{ab}] & \text{if } a \leq c, b \leq d; \\ [e_{cd} \otimes e_{ab}] + (q-1)[e_{ab} \otimes e_{cd}] & \text{if } a > c, b > d; \\ [e_{cd} \otimes e_{ab}] + (q-1)[e_{ad} \otimes e_{cb}] & \text{if } a > c, b \leq d; \\ q[e_{cd} \otimes e_{ab}] + (q-1)[e_{ab} \otimes e_{cd}] - (q-1)[e_{ad} \otimes e_{cb}] & \text{if } a \leq c, b > d. \end{cases}$$

As usual, we identify q with v^2. It remains to be seen that this action is well-defined.

The action of $\mathcal{H}(A_{r-1})$ on the quantized basis elements of $T^r(M_n)$ is defined entirely analogously, with a typical generator $T_{(p,p+1)}$ acting as in the $r = 2$ case, but affecting only the p-th and $(p+1)$-th positions in the tensor.

Lemma 3.1 The above action is a well-defined action of $\mathcal{H}(A_{r-1})$ on $T^r(M_n)$.

Proof It is well-known (see for example [**8**]) that $\mathcal{H}$ is given by generators $T_1, \ldots, T_{r-1}$ subject to the following relations:

$$T_s^2 = qT_e + (q-1)T_s,$$

$$T_sT_t = T_tT_s \quad \text{if } |s-t| > 1,$$

$$T_sT_tT_s = T_tT_sT_t \quad \text{if } |s-t| = 1.$$

Here, T_e is the multiplicative identity in $\mathcal{H}$, and s and t are integers between 1 and $r-1$ inclusive.

To check the first of these relations, it is enough to consider the case $r = 2$. The element T_s acts on the space spanned by the ordered basis

$$[e_{ab} \otimes e_{cd}], [e_{cd} \otimes e_{ab}], [e_{cb} \otimes e_{ad}], [e_{ad} \otimes e_{cb}],$$

where we assume that $a < c$ and $b < d$. With respect to this basis, the right action of T_s is given by the matrix

$$M := \begin{pmatrix} 0 & q & 0 & 0 \\ 1 & q-1 & 0 & 0 \\ 0 & q-1 & 0 & 1 \\ -(q-1) & 0 & q & q-1 \end{pmatrix}.$$

It is routine to check that $M^2 = qI + (q-1)M$, where I is the identity matrix. There are also three degenerate cases to check.

1. The case $a = c, b < d$. Consider the basis $[e_{ab} \otimes e_{ad}], [e_{ad} \otimes e_{ab}]$. Using the previous method, we find the action of T_s is given by

   $$M := \begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix}.$$

   This matrix satisfies $M^2 = qI + (q-1)M$ as required.

2. The case $a < c, b = d$. This is very similar to case 1., except that the basis to take is $[e_{ab} \otimes e_{cb}], [e_{cb} \otimes e_{ab}]$.

3. The case $a = c, b = d$. Here, T_s acts on $[e_{ab} \otimes e_{ab}]$ via the ind representation, i.e. $T_s \mapsto q$. This satisfies the required relation.

Since $T^r(M_n)$ is spanned by quantized basis elements, each of which falls into exactly one of the above four classes, we have shown that the first relation, $T_s^2 = qT_e + (q-1)T_s$, always holds.

The second relation (known as a short braid relation), $T_sT_t = T_tT_s$ when $|s-t| > 1$, is an easy case. The action of $T_{(p,p+1)}$ is a sum of certain permutations of coordinates in the matrix units occuring in the p-th and $(p+1)$-th places, so it is clear that if $|s-t| > 1$, the permutations will not interfere with each other, and hence the elements T_s and T_t will commute with each other.

The last relation (known as a long braid relation) is the trickiest to check. To help in the checking, we recall the action of $\mathcal{H}$ on tensor space, as defined by Du [**5**] and others.

Let V be, as usual, a vector space over $\mathbf{Q}(v)$ with basis $e_1, \ldots, e_n$. We now define quantized basis elements of $T^r(V)$ from the usual basis as follows:

$$[e_{i_1} \otimes \cdots \otimes e_{i_r}] := v^{-m} e_{i_1} \otimes \cdots \otimes e_{i_r},$$

where

$$m := |\{(a,b) : 1 \leq a < b \leq r,\ i_a > i_b\}|.$$

Following [**5**], we see that the action of $\mathcal{H}$ on $T^r(V)$ is as follows:

$$[e_{i_1} \otimes \cdots \otimes e_{i_r}].T_{p,p+1} = \begin{cases} q[e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r}] & \text{if } i_p \leq i_{p+1}; \\ [e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r}] + (q-1)[e_{i_1} \otimes \cdots \otimes e_{i_r}] & \text{if } i_p > i_{p+1}. \end{cases}$$

We will exploit the fact that this action resembles the case $b \leq d$ in our candidate for the action of $\mathcal{H}$ on $T^r(M_n)$.

To deal with the case $b > d$, we calculate the action of T_s^{-1} in each case. This follows easily from the fact that $T_s^{-1} = (q^{-1} - 1)T_e + q^{-1}T_s$ and the original definitions, because T_e acts as the identity. The action of T_s^{-1} on $T^2(M_n)$ is as follows:

$$[e_{ab} \otimes e_{cd}].T_s^{-1} := \begin{cases} q^{-1}[e_{cd} \otimes e_{ab}] & \text{if } a \geq c, b \geq d; \\ [e_{cd} \otimes e_{ab}] + (q^{-1}-1)[e_{ab} \otimes e_{cd}] & \text{if } a < c, b < d; \\ [e_{cd} \otimes e_{ab}] + (q^{-1}-1)[e_{ad} \otimes e_{cb}] & \text{if } a < c, b \geq d; \\ q^{-1}[e_{cd} \otimes e_{ab}] + (q^{-1}-1)[e_{ab} \otimes e_{cd}] - (q^{-1}-1)[e_{ad} \otimes e_{cb}] & \text{if } a \geq c, b < d. \end{cases}$$

The action of $T_{(p,p+1)}^{-1}$ on $T^r(V)$ is given as follows:

$$[e_{i_1} \otimes \cdots \otimes e_{i_r}].T_{p,p+1}^{-1} = \begin{cases} q^{-1}[e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r}] & \text{if } i_p \geq i_{p+1}; \\ [e_{i_1} \otimes \cdots \otimes e_{i_{p+1}} \otimes e_{i_p} \otimes \cdots \otimes e_{i_r}] + (q^{-1}-1)[e_{i_1} \otimes \cdots \otimes e_{i_r}] & \text{if } i_p < i_{p+1}. \end{cases}$$

Note the similarity between the action on tensor space and the $b \geq d$ case in the prospective matrix action here. Also note that all the actions have a "continuity" property: for example in any case where $a = c$, it does not matter whether this is considered as $a \leq c$ or $a \geq c$.

Comparing the actions of T_s on $[e_{ab} \otimes e_{cd}]$ and on $[e_a \otimes e_c]$, we find that they are very similar, provided that $b \leq d$. What happens is that the second coordinates in the matrix units are exchanged, and the action on the first coordinates is exactly the same as in the action on tensor space. On the other hand, the action of T_s^{-1} on $[e_{ab} \otimes e_{cd}]$ is similar to the action of T_s^{-1} on $[e_a \otimes e_c]$, provided that $b \geq d$. Again, the second coordinates are exchanged, and the first coordinates are acted upon as in the action on tensor space.

With these observations out of the way, we proceed with the proof. Clearly, it is enough to study the case $r = 3$. Pick a quantized standard basis element, $[e_{ab} \otimes e_{cd} \otimes e_{ef}]$. It is enough to show that $T_s T_t T_s$ acts in the same way as $T_t T_s T_t$ on this basis element, where s corresponds to the permutation $(1, 2)$ and t corresponds to the permutation $(2, 3)$. The above discussion shows that this is guaranteed, provided $b \leq d \leq f$, because we know that the action on tensor space is well-defined. For the other cases, we need another method. For example, if $b > d > f$, we know from the discussion that $T_s^{-1} T_t^{-1} T_s^{-1}$ acts in the same way as $T_t^{-1} T_s^{-1} T_t^{-1}$, and this is equivalent to showing that the long braid relation holds. There are six possible cases. Here is a table showing the correct method of proof in each case. (There is some overlap—in these cases, any of the given methods will work.) For clarity, σ means T_s and τ means T_t.

Situation	Method of Proof
$b \leq d \leq f$	$\sigma\tau\sigma = \tau\sigma\tau$
$f \leq d \leq b$	$\sigma^{-1}\tau^{-1}\sigma^{-1} = \tau^{-1}\sigma^{-1}\tau^{-1}$
$d \leq b \leq f$	$\sigma^{-1}\tau\sigma = \tau\sigma\tau^{-1}$
$d \leq f \leq b$	$\sigma^{-1}\tau^{-1}\sigma = \tau\sigma^{-1}\tau^{-1}$
$b \leq f \leq d$	$\sigma\tau\sigma^{-1} = \tau^{-1}\sigma\tau$
$f \leq b \leq d$	$\sigma\tau^{-1}\sigma^{-1} = \tau^{-1}\sigma^{-1}\tau$

This suffices to prove that the long braid relation is preserved.

This completes the proof. ∎

It will be important in the proofs which follow that the actions of $U(gl_n)$ and $\mathcal{H}(A_{r-1})$ on $T^r(M_n)$ commute with each other. The action of U on M_n is given by:

$$E_i e_{jk} := \delta_{i,j-1} e_{j-1k},$$

$$F_i e_{jk} := \delta_{i,j} e_{j+1k},$$

$$K_i e_{jk} := v^{\delta_{i,j}} e_{jk},$$

$$K_i e_{jk} := v^{-\delta_{i,j}} e_{jk},$$

where δ is the Kronecker delta. The action extends to $T^r(M_n)$ via $\Delta^{(r-1)}$.

Lemma 3.2 The actions of $\mathcal{H}(A_{r-1})$ and U on $T^r(M_n)$ commute with each other.

Proof We will rely heavily on the well-known fact that the actions of U and $\mathcal{H}$ on tensor space commute with each other. Note that our action of U on $T^r(M_n)$ is essentially the action of U on tensor space, but ignoring the second coordinate.

Clearly the action of U commutes with the action of T_e. It is now enough, for each basis element of $T^r(M_n)$ and for each Hecke algebra generator T_s (corresponding to a simple reflection in the Coxeter group), to establish either that the action of each E_i, F_i, K_i and K_i^{-1} commutes with the action of T_s *or* the action of T_s^{-1} (because $T_s^{-1} = (q^{-1} - 1)T_e + q^{-1}T_s$). A moment's thought shows that it is enough to consider the case $r = 2$, because only simple transpositions are involved.

Denote the typical basis element by $e_{ab} \otimes e_{cd}$ (note that we choose to work with the unquantized basis of $T^r(M_n)$). We first dispose of the easy cases of the K_i and K_i^{-1}. The action of K_i on the basis element is to multiply by v^m, where m is the number of first coordinates in the basis element equal to i. The family of first coordinates is unchanged under the action of $\mathcal{H}$, so the actions commute. The case of K_i^{-1} is similar, but with v^{-m} in place of v^m.

Now we consider the case of E_i. Assume firstly that $b \leq d$. We wish to show that the actions of E_i and T_s commute. However, we know from the proof of Lemma 3.1 that the action of T_s on $T^r(M_n)$, when projected to the first coordinates, is similar to the action of T_s on $T^r(V)$, except that the two second coordinates are exchanged and the expression is multiplied by v^{-1} (if $b < d$) or v^0 (if $b = d$), since we are working with unquantized basis elements. We know that the actions of E_i and T_s on tensor space commute. Because the action of U does not change second coordinates of matrix units, we see that the actions of E_i and T_s do indeed commute. Now assume that $b > d$. A similar argument now applies to the actions of E_i and T_s^{-1}. (Here, the second coordinates are exchanged by T_s^{-1} and the expression is multiplied by v.) Hence the actions of E_i and T_s^{-1} commute.

The same techniques as we used for E_i can be used to show that the action of F_i commutes with the action of $\mathcal{H}$.

This completes the proof. ∎

We are now ready to define the candidate for the monomorphism γ of algebras and U-modules from $S_v(n,r)$ into $T^r(M_n)$. Most of the rest of this paper will be devoted to proving that γ satisfies the required properties.

Recall that $S_v(n,r)$ has a basis denoted by $\{[A]\}$, as A ran over all $n \times n$ matrices with nonnegative integer entries summing to r. With each matrix A, we can (uniquely) associate a standard basis element t_A of $T^r(M_n)$, given by

$$e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r},$$

exactly as in the classical case. Define $\mathcal{D}_A$ as in the classical case also. Define the integer m_A by

$$m_A := |\{(i,j) : 1 \leq i < j \leq r,\ b_i = b_j,\ a_i < a_j\}|.$$

Example Let A be the matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then t_A is given by

$$e_{11} \otimes e_{11} \otimes e_{11} \otimes e_{21} \otimes e_{12} \otimes e_{22} \otimes e_{22},$$

and $m_A = 5$.

Definition

The linear map $\gamma : S_v(n, r) \rightarrow T^r(M_n)$ is given by

$$\gamma([A]) := v^{-m_A} t_A. \left(\sum_{w \in \mathcal{D}_A} T_w \right),$$

where the action of $\mathcal{H}(A_{r-1})$ on $T^r(M_n)$ is as before.

Note Using the map c appearing in the proof of Proposition 2.2, we can regard γ as a map from $S_v(n, r)$ to $\mathrm{End}(V^{\otimes r})$.

Lemma 3.3

a) Let

$$t_A := e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r},$$

and let

$$t'_A := e_{c_1 d_1} \otimes \cdots \otimes e_{c_r d_r}$$

be a standard basis element of $T^r(M_n)$ such that $d_i = b_i$ for all i and the ordered pairs (c_i, d_i) are a rearrangement of the ordered pairs (a_i, b_i). Then the coefficient of t'_A in $\gamma([A])$ is $v^{-m'}$, where

$$m' := |\{(i, j) : 1 \leq i < j \leq r,\ d_i = d_j,\ c_i < c_j\}|.$$

b) Any basis element $t = e_{x_1 b_1} \otimes \cdots \otimes e_{x_r b_r}$ occurring with nonzero coefficient in $\gamma([A])$ must satisfy the hypotheses satisfied by t'_A above.

Proof We deal with the proof of a) first.

Firstly, we consider the action of $\sum_{w \in \mathcal{D}_A} T_w$ on $[t_A]$, which of course differs from t_A by some power of v. Consider the action of a typical T_w in the sum, and write

$$w = s_{i_1} \cdots s_{i_m}$$

as a reduced product of simple reflections.

We revert for a moment to acting $\mathcal{S}_r$ on $T^r(M_n)$. Because w is a distinguished coset representative, we find that the matrix units occurring at positions i_j and i_{j+1} in

$$t_j := t_A.s_{i_1}\cdots s_{i_{j-1}},$$

will be in the correct order with respect to the Hebrew lexicographic order (as defined in §1.7). (This follows from the fact that $\ell(s_{i_1}\cdots s_{i_{j-1}}s_{i_j}) = \ell(s_{i_1}\cdots s_{i_{j-1}})+1$, and standard properties of the length function.) In particular, if

$$t_j = v^* e_{a_1 b_1}\otimes\cdots\otimes e_{a_r b_r},$$

we have $b_{i_j} \leq b_{i_{j+1}}$. (Here, "$*$" denotes some integer whose value is unimportant.) Because of the above remark concerning the Hebrew lexicographic order, we deduce that when computing $t_A.T_w$, we never have to appeal to the case corresponding to $b > d$ in the definition of the action of T_s.

This has repercussions for the quantized case. Recall the remarks in the proof of 3.1 about the similarities between the actions of T_s on tensor space and tensor matrix space, in the case $b \leq d$—the action on the second coordinates is essentially permutation. Taking the coefficient of a typical quantized basis element appearing in $[t_A].T_w$, say

$$t := e_{c_1 d_1}\otimes\cdots\otimes e_{c_r d_r},$$

we now know from the above discussion that if

$$t_A := e_{a_1 b_1}\otimes\cdots\otimes e_{a_r b_r}$$

then the sequence $(d_1,\ldots,d_r)$ is simply the result of acting w on the right of the sequence $(b_1,\ldots,b_r)$ by place permutation.

Let W_J be the parabolic subgroup of $W(A_{r-1})$ which is generated by the $(p,p+1)$ satisfying $b_p = b_{p+1}$. We then see from the above discussion that unless $w \in W_J$, there will certainly be no nonzero coefficient of t'_A occurring in $[t_A].T_w$ (because no basis elements occurring with nonzero coefficient in the latter expression can have their second coordinates in the correct order), and hence in $t_A.T_w$. This simplifies the problem considerably.

The next important observation is that, if $a < b$,

$$e_{ac}\otimes e_{bc}.T_s = v e_{bc}\otimes e_{ac}.$$

Informally, every time an element T_s (where $s \in W_J$) exchanges two matrix units out of Hebrew lexicographic order, the exponent of v is increased by 1. We know that there exists a unique element $u \in W_J \cap \mathcal{D}_A$ such that $t'_A = t_A.u$ (acting as an element of the symmetric group). (Conversely, for any element $u' \in W_J \cap \mathcal{D}_A$, there is a corresponding unique t''_A, with properties similar to those of t'_A.) Write u as a reduced product of simple reflections, $s_{i_1} \cdots s_{i_l}$. Then the entries in the i_k and i_{k+1}-th positions of $t_A.s_{i_1} \cdots s_{i_{k+1}}$ are in Hebrew lexicographic order with different first coordinates and equal second coordinates. Returning to the quantized case (which essentially differs only by powers of v), let the coefficient of t_A in $\gamma([A])$ be C. The number of pairs of matrix units exchanged out of order in passing from t_A to t'_A is $m_A - m'$, where m' is as in the statement of the Lemma. Therefore, we find that the coefficient of t'_A in $C.t_A.\sum_{w \in W_J \cap \mathcal{D}_A} T_w$, and hence in $\gamma([A])$, is $C.v^{m_A - m'}$, by using the above observations. However, we know from the definition of γ (and the correspondence between u' and t''_A mentioned above, in the case where $u' = 1$ and $t''_A = t_A$) that $C = v^{-m_A}$. Thus the coefficient of t'_A is $v^{-m'}$ as claimed.

The proof of b) also follows from the preceding arguments. Let $t = e_{x_1 b_1} \otimes \cdots \otimes e_{x_r b_r}$. We know that the only terms of $\gamma([A])$ which can yield this are of form $t_A.T_w$, where $w \in W_J \cap \mathcal{D}_A$. We have also shown that for such an element w, $t_A.T_w$ is equal to

$$v^* e_{y_1 b_1} \otimes \cdots \otimes e_{y_r b_r},$$

where the (y_i, b_i) are some rearrangement of the original (a_i, b_i). This completes the proof of b). ∎

Before proceeding with the main theorem, it is worth defining the *relaxed* order, $\preceq$, on the matrix units. We say that $e_{ab} \preceq e_{cd}$ if ($b < d$) or ($b = d$ and $a > c$) or ($b = d$ and $a = c$). A basis element of $T^r(M_n)$ is said to be *relaxed* if the matrix units yielding it appear in it in relaxed order. A basis element is called *H-relaxed* if the matrix units in it appear in Hebrew lexicographic order (for example, t_A for any basis matrix A).

Example The basis element of $T^r(M_n)$ given by

$$e_{21} \otimes e_{11} \otimes e_{11} \otimes e_{11} \otimes e_{22} \otimes e_{22} \otimes e_{12}$$

is relaxed.

One of the main points of Lemma 3.3 is that it gives the following result.

Corollary 3.4

(i) In the expression $\gamma([A])$, with respect to the standard basis of $T^r(M_n)$, the basis element R_A of $T^r(M_n)$ which is the relaxed rearrangement of t_A is the only relaxed basis element appearing with nonzero coefficient. Furthermore, it appears with coefficient 1.

(ii) In the expression $\gamma([A])$, with respect to the standard basis of $T^r(M_n)$, the basis element t_A is the only H-relaxed basis element appearing with nonzero coefficient. It appears with coefficient v^{-m_A}.

Proof Note that the element R_A is given by

$$\underbrace{e_{n1} \otimes \cdots \otimes e_{n1}}_{A_{n1}} \otimes \cdots \otimes \underbrace{e_{11} \otimes \cdots \otimes e_{11}}_{A_{11}} \otimes \underbrace{e_{n2} \otimes \cdots \otimes e_{n2}}_{A_{n2}} \otimes \cdots \otimes \underbrace{e_{12} \otimes \cdots \otimes e_{12}}_{A_{12}} \otimes \cdots .$$

By definition, any relaxed basis element, R, appearing must have its second coordinates appearing in ascending order. Using Lemma 3.3, R must therefore be obtainable from t_A by action on the right by an element $w \in W_J \cap \mathcal{D}_A$, in the notation of the proof of 3.3. This means that R satisfies the hypotheses for t'_A in the statement of Lemma 3.3. Since R is relaxed, it must be equal to R_A (there is no alternative). We then find that the integer m' defined in the Lemma is equal to 0, thus showing that R_A appears with coefficient 1.

The proof of (ii) uses the same techniques. In this case, $w = 1$ and $m' = m_A$. ∎

Example Let A be the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $R_A = e_{21} \otimes e_{12}$ and

$$\gamma([A]) = e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + (v^2 - 1)e_{22} \otimes e_{11}.$$

Note that the coefficient of R_A in $\gamma([A])$ is 1.

We are now ready to show that γ is an isomorphism of U-modules.

Lemma 3.5

(i) The images of the basis $\{[A]\}$ of $S_v(n, r)$ under γ span a U-submodule of $T^r(M_n)$.

(ii) The U-submodule defined above is also an $\mathcal{H}$-submodule, and is the largest $\mathcal{H}$-submodule of $T^r(M_n)$ on which T_w acts as the scalar $q^{\ell(w)}$. (We will refer to this $\mathcal{H}$-submodule as the *ind submodule.*)

(iii) The map γ is an isomorphism of U-modules between $S_v(n, r)$ and the submodule described in (i).

Proof First, we compare $\gamma([A])$ with the element

$$g([A]) := t_A . \left(\sum_{w \in W(A_{r-1})} T_w \right).$$

Using the standard theory of Coxeter groups and Hecke algebras, and denoting by W_A the parabolic subgroup corresponding to $\mathcal{D}_A$, we can rewrite $g([A])$ as

$$t_A . \left(\sum_{u \in W_A} T_u \right) \left(\sum_{w \in \mathcal{D}_A} T_w \right).$$

The point of doing this is that the action of T_u, where $u \in W_A$, on t_A is very simple—it simply multiplies by the scalar $q^{\ell(u)}$. We know (see for example, [**8**, §3.15]) that the Poincaré polynomial of a Coxeter group of type A_l is

$$\prod_{i=1}^{l} \sum_{k=0}^{i} t^k,$$

and we also know that W_A, being a parabolic subgroup of a Coxeter group of type A, is canonically isomorphic to a direct product of such groups of smaller rank. Its Poincaré polynomial is therefore a product of expressions such as those given above. By substituting $t = q$, we obtain the scalar by which $\sum_{u \in W_A} T_u$ acts, and since $1 + q + \cdots + q^{e-1} \neq 0$ for any natural number e and $\mathbf{Q}(v)$ is a field, we find that this scalar is nonzero. Bringing the quantity v^{-m_A} back into the picture, we find that $\gamma([A])$ and $g([A])$ differ only by multiplication by a nonzero scalar. To prove (i), it is therefore enough to prove that the span of elements $g([A])$ is fixed setwise under the action of U.

Before completing the proof of (i), it is convenient to prove (ii). It is well known and easy to show that

$$\left(\sum_{w \in W} T_w \right) T_d = q^{\ell(d)} \left(\sum_{w \in W} T_w \right). \tag{1}$$

This means that

$$g([A]).T_d = q^{\ell(d)} g([A]),$$

by Lemma 3.1, and hence $g([A])$ lies in the ind submodule. We claim that any $t \in T^r(M_n)$ satisfying

$$t.T_w = q^{\ell(w)} t$$

for all $w \in W(A_{r-1})$ lies in the span of the elements $g([A])$. It is easy to see that

$$t = c.t \left(\sum_{w \in W(A_{r-1})} T_w \right)$$

for a certain scalar $c \neq 0$. Let $t_1 = e_{a_1 b_1} \otimes \cdots \otimes e_{a_r b_r}$ be a basis element occurring in t. If we can prove that

$$t_1. \left(\sum_{w \in W(A_{r-1})} T_w \right)$$

lies in the span of the $g([A])$ then the proof of (ii) will follow.

We first deal with the case where $b_1 \leq \cdots \leq b_r$. The proof of Lemma 3.3 now shows that $t_1 = v^* t_A.T_w$ for some H-relaxed basis element t_A and some $w \in W(A_{r-1})$. The assertion now follows by using (1). In the case where the sequence of b_i is not increasing, let $(d_1, \ldots, d_r)$ be the rearrangement of $(b_1, \ldots, b_r)$ satisfying $d_1 \leq \cdots \leq d_r$. Choose w such that

$$(d_1, \ldots, d_r).w = (b_1, \cdots, b_r).$$

A simple inductive argument like the one in the proof of Lemma 3.3 shows that all elements occurring with nonzero coefficient in $t_1.T_w^{-1}$ are of form

$$e_{x_1 d_1} \otimes \cdots \otimes e_{x_r d_r}.$$

We now use (1) and the previous case (where $b_1 \leq \cdots \leq b_r$) to deduce the required result, thus completing the proof of (ii).

We return to the proof of (i). Let $u \in U$. We know from Lemma 3.2 that

$$u.\left(t_A.\sum_{w \in W(A_{r-1})} T_w\right) = (u.t_A).\sum_{w \in W(A_{r-1})} T_w,$$

and this latter element clearly lies in the ind submodule, and hence in the span of the elements $g([A])$. This completes the proof of (i).

For the proof of (iii), we compare the results we get by acting U on $\gamma(S_v(n,r))$ with those in [**1**, Lemma 3.4]. It is enough to verify that the action of u on $\gamma([A])$ agrees with the results of [**1**, Lemma 3.4] where $u \in \{E_*, F_*, K_*, K_*^{-1}\}$, $[A]$ is a basis element for $S_v(n,r)$, and $u \in U$ acts on $[A] \in S_v(n,r)$ via $u.[A] := \theta(u) \times [A]$. The results we are hoping for are:

a)

$$K_h.\gamma([A]) = v^{s_h}\gamma([A]),$$

where $s_h := \sum_{k=1}^n A_{hk}$.

$$K_h^{-1}.\gamma([A]) = v^{-s_h}\gamma([A]),$$

where $s_h := \sum_{k=1}^n A_{hk}$.

b)

$$E_h.\gamma([A]) = \sum_{p \in [1,n]: A_{h+1,p} \geq 1} v^{\beta(p)}[A_{h,p}+1]\ \gamma\left([A + E_{h,p} - E_{h+1,p}]\right),$$

where

$$\beta(p) := \sum_{j>p}(A_{h,j} - A_{h+1,j}).$$

c)

$$F_h.\gamma([A]) = \sum_{p \in [1,n]: A_{h,p} \geq 1} v^{\beta'(p)}[A_{h+1,p}+1]\ \gamma\left([A + E_{h+1,p} - E_{h,p}]\right),$$

where

$$\beta'(p) := \sum_{j<p}(A_{h+1,j} - A_{h,j}).$$

Case a) is easy to verify, because

$$K_h.e_{c_1 d_1} \otimes \cdots \otimes e_{c_r d_r} := v^s e_{c_1 d_1} \otimes \cdots \otimes e_{c_r d_r},$$

where s is the number of c_i equal to h. By checking coefficients of relaxed terms, via Corollary 3.4, we see that the first half of a) holds. Similar techniques work for K_h^{-1}.

To prove case b), choose a basis element $[B]$ which occurs in $E_h.[A]$. It is enough to check that the coefficient is correct. Because of the way E_h acts on $T^r(M_n)$, we can find a close relationship between the relaxed basis elements R_B and R_A. By using Lemma 3.3 b), we see that the only difference is that the rightmost occurrence of $e_{h+1,p}$ in R_A is replaced by $e_{h,p}$ to obtain R_B, for some integer $p : 1 \leq p \leq n$. (If this cannot happen, then the coefficient must be zero as desired.) Let us say that this difference occurs between the P-th component of the basis elements.

We need to identify all basis elements occurring with nonzero coefficients in $\gamma([A])$ whose images under the action of E_h contain a nonzero coefficient of R_B. We do this by considering the action of E_h on $T^r(M_n)$. An example of such an element is R_A itself. A typical basis element R satisfying these hypotheses differs from R_A just to the right of the P-th component; where R_A contains the sequence

$$\cdots \otimes e_{h+1,p} \otimes e_{h,p} \otimes \cdots \otimes e_{h,p} \otimes \cdots,$$

R contains some permutation of this sequence. (Informally, $e_{h+1,p}$ may occur a few places to the right of where it "should".) Denote by R_z the basis element R (as above) where the rightmost occurrence of $e_{h+1,p}$ occurs at place $P+z$ (so in particular, $R_0 = R_A$). Let $h(z)$ denote the multiplicity of $e_{h,p}$ in R_A. (It is equal to $A_{h,p}$.)

We know from Lemma 3.3 that R_z occurs in $\gamma([A])$ with coefficient v^{-z}.

It can be shown that

$$\Delta^{(r-1)}(E_h) = 1 \otimes \cdots \otimes E_h + 1 \otimes \cdots \otimes E_h \otimes K_i K_{i+1}^{-1} + \cdots + E_h \otimes K_i K_{i+1}^{-1} \otimes \cdots \otimes K_i K_{i+1}^{-1},$$

where there are r tensor components in each summand. Using this fact, we find that the contribution of $v^{-z} R_z$ to the coefficient of R_B in the result is v^{x_z}, where

$$x_z := -z + h(z) - z + \left(\sum_{p'>p} A_{h,p'} \right) - \left(\sum_{p'>p} A_{h+1,p'} \right).$$

Summing over all possible z (i.e. $0 \leq z \leq h(z)$), we find that the total contribution from $\gamma([A])$ to R_B is $v^{\beta(p)}[A_{h,p}+1]$, as in the statement of the Lemma.

Case c) is very similar to case b), so we omit the proof.

This completes the proof of the Lemma. ∎

Now that we know that γ is an isomorphism of U-modules, it is not too hard to show that it is an isomorphism of algebras. We know that $S_v(n,r)$ is generated as an algebra by $\theta(E_h), \theta(F_h), \theta(K_i), \theta(K_i^{-1})$, where $1 \leq h < n$ and $1 \leq i \leq n$. It therefore suffices to show that

$$\gamma(\theta(u) \times [A]) = \gamma(\theta(u)) \times \gamma([A]),$$

for all $[A]$ and for all u in the standard algebra generating set $\{E_*, F_*, K_*, K_*^{-1}\}$ of U.

Theorem 3.6 The map γ is a canonical monomorphism of algebras and of U-modules from $S_v(n,r)$ to $T^r(M_n)$.

Proof Recall the remark after Proposition 2.3. Again, it is true that the action of $u \in U$ on an element $t \in T^r(M_n)$ is given by

$$u.t = \gamma_r(u) \times t,$$

using the quantized version of the map γ_r appearing in Proposition 2.3. We claim that it is enough to show that $\gamma_r(u) = \gamma(\theta(u))$ for elements u in the standard generating set. For if this is true, Lemma 3.5, the preceding remark, and the fact that

$$u.[A] := \theta(u) \times [A]$$

show that

$$\gamma(\theta(u) \times [A]) = \gamma(u.[A]) = u.\gamma([A]) = \gamma_r(u) \times \gamma([A]) = \gamma(\theta(u)) \times \gamma([A]),$$

as required.

The case $u = K_i$ is fairly easy, so we tackle this first. Beilinson et al. [**1**] define

$$\theta(K_i) = \sum_{D \in \mathbf{D}_r} v^{d_i}[D],$$

where $\mathbf{D}_r$ is the set of $n \times n$ diagonal matrices with nonnegative integer entries summing to r. Calculating $\gamma([D])$, where D is a diagonal matrix, turns out to be remarkably simple, because $m_D = 0$ and because the Hecke algebra $\mathcal{H}(A_{r-1})$ acts via permutations! The reason for the latter is as follows. Consider a typical element $T_w \in \mathcal{D}_D$, and write $w = s_{i_1} \cdots s_{i_l}$ as a reduced expression, as usual. Let

$$t_j := t_D.s_{i_1} \cdots s_{i_{j-1}}.$$

As in the proof of Lemma 3.3, the matrix units occurring at positions i_j and i_{j+1} in t_j are lexicographically ordered. However, if $a < b$,

$$e_{aa} \otimes e_{bb}.T_s = e_{bb} \otimes e_{aa},$$

so we see that $\mathcal{H}$ acts only via permutations. From this, we see that

$$\gamma(\theta(K_i)) = \sum_D \left(v^{d_i}.t_D.\left(\sum_{w \in \mathcal{D}_D} T_w \right) \right),$$

where the outer sum ranges over all possible values of the diagonal matrices $D = \mathrm{diag}(d_1, \ldots, d_r)$ subject to the constraint $\sum d_i = r$.

We now check that this is equal to $\gamma_r(K_i)$. It is not hard to show that

$$\Delta^{(r-1)}(K_i) = \underbrace{K_i \otimes \cdots \otimes K_i}_{r \text{ times}}.$$

We know that

$$\theta_1(K_i) = \sum_{j=1}^{n} v^{\delta(i,j)} e_{j,j}.$$

Using the fact that

$$\gamma_r := \underbrace{(\theta_1 \otimes \cdots \otimes \theta_1)}_{r \text{ times}} \Delta^{(r-1)},$$

it is fairly easy to see that the maps γ_r and $\gamma\theta$ agree on K_i, as required.

The case $u = K_i^{-1}$ follows by an argument entirely analogous to the previous one.

We now tackle the case of $u = E_h$. Following [**1**, §5.7], we have

$$\theta(E_h) := \sum_{D \in \mathbf{D}_{r-1}} [E_{h,h+1} + D].$$

We then use a similar technique to the one used in the K_i case to argue that $\mathcal{H}$ "almost" acts via permutations (more precisely—it does act via permutations, but introduces certain powers of v). The situation is very similar to the K_i situation, *except* that

$$e_{hh+1} \otimes e_{h+1h+1}.T_s = v e_{h+1h+1} \otimes e_{hh+1}$$

and

$$e_{hh} \otimes e_{hh+1}.T_s = v e_{hh+1} \otimes e_{hh}.$$

Informally, this means that the situation is similar to the classical case, except the processes of passing an e_{h+1h+1} to the left of the e_{hh+1} and that of passing an e_{hh} to the right of an e_{hh+1} multiply the relevant basis element by v. However, we know from Corollary 3.4 that "relaxed" basis elements have coefficient 1. Such basis elements contain $e_{h,h+1}$ to the right of any possible occurrences of e_{hh} and e_{h+1h+1}. This means that the coefficient of any typical basis element of $T^r(M_n)$ occurring in $\gamma(\theta(E_h))$, say

$$t = e_{c_1 d_1} \otimes \cdots \otimes e_{c_r d_r},$$

occurs with coefficient v^s, where s is the number of e_{hh} occurring to the right of the occurrence of e_{hh+1} minus the number of e_{h+1h+1} occurring to the right of the occurrence of e_{hh+1}.

We know that

$$\Delta^{(r-1)}(E_h) = 1 \otimes \cdots \otimes E_h + 1 \otimes \cdots \otimes E_h \otimes K_i K_{i+1}^{-1} + \cdots + E_h \otimes K_i K_{i+1}^{-1} \otimes \cdots \otimes K_i K_{i+1}^{-1},$$

where there are r tensor components in each summand. Since $\theta_1(E_h) = e_{hh+1}$ and we know what the $\theta_1(K_i^{\pm 1})$ are, it is possible to see from the above discussion that the maps γ_r and $\gamma\theta$ agree on E_h.

The case of $u = F_h$ is entirely analogous to that of $u = E_h$.

This completes the proof of the main result. ∎

Example Consider $S_v(2,2)$. Let

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let I be the 2×2 identity matrix. Let e_A and e_I be the corresponding elements in the Dipper-James basis of $S_v(2,2)$. We know from properties of the Hecke algebra of type A_1 embedded in $S_v(2,2)$ that $e_A^2 = qe_I + (q-1)e_A$, but we wish to show this directly. It can be checked that $v[A] = e_A$ and $[I] = e_I$.

Calculating the values of γ on these basis elements, we find that

$$v.\gamma([A]) = ve_{12} \otimes e_{21} + ve_{21} \otimes e_{12} + (v^2-1)e_{22} \otimes e_{11},$$

and that

$$\gamma([I]) = e_{11} \otimes e_{22} + e_{22} \otimes e_{11}.$$

Squaring $v.\gamma([A])$ in the ring $T^r(M_n)$, we find that it gives

$$v^2 e_{11} \otimes e_{22} + v^2 e_{22} \otimes e_{11} + (v^2-1)^2 e_{22} \otimes e_{11} + (v^3 - v)e_{21} \otimes e_{12} + (v^3 - v)e_{12} \otimes e_{21}.$$

The relaxed basis elements occurring in these expressions are $e_{11} \otimes e_{22}$, corresponding to $[I]$, and occurring with coefficient $q = v^2$, and $e_{21} \otimes e_{12}$, corresponding to $[A]$ and occurring with coefficient $(v^3 - v) = (q-1)v$. Hence we have

$$(v[A])^2 = q[I] + (q-1)(v[A]),$$

thus proving the desired result. ∎

Remarks on Theorem 3.6 As can be seen from the example, these methods give a fairly easy way of computing products in $S_v(n,r)$, without having to resort to difficult procedures involving flags and finite fields.

Since the classical case corresponding to this (i.e. when q and v are replaced by 1) gives the situation where the classical Schur algebra is the r-th symmetric power of the $n \times n$ matrix algebra, the Theorem casts more light on the nature of the quantization. The space $T^r(M_n)$ provides an arena in which the quantization takes place.

It should be noted that the theorem has can easily be adapted to apply $S_q(n,r)$ in terms of the more usual basis $\{e_A\}$. In [**1**, Lemma 2.3], it is proved that $e_A = v^{s_A}[A]$, where

$$s_A := \sum_{1 \leq i,j,k,l \leq n} A_{i,j} A_{k,l},$$

where the sum is taken with the restrictions that $i \geq k$ and $j < l$. This exhibits a close connection between the two bases. The restrictions on the sum are reminiscent of the relaxed order, $\preceq$, defined on the matrix units.

The above theorem is convenient for proving results whose classical analogues are proved using Schur's Product Rule. In a future paper, the author will use the theorem to prove a quantum analogue of Woodcock's Straightening Formula for Codeterminants, which was described in [**12**].

We now remark that the results of §3 are compatible with the results of Dipper and James [**3**] on q-tensor space.

Proposition 3.7 Suppose $n \geq r$. Regard γ, via the canonical isomorphism $c : \mathrm{End}(V^{\otimes r}) \cong T^r(M_n)$, as a map from $S_v(n,r)$ to $\mathrm{End}(V^{\otimes r})$. Then the resulting action of $S_v(n,r)$ on $V^{\otimes r}$ is canonically equivalent to the Dipper–James action of $S_q(n,r)$ on "q-tensor space".

Proof Essentially this is the observation, made in [**5**, §1.5] that the q-tensor space of [**3**] and $V^{\otimes r}$ are isomorphic as U–$\mathcal{H}$ bimodules; the isomorphism is given by

$$v^{\#\{(a,b):\ a<b,\ i_a<i_b\}} e_{(i_1,\ldots,i_r)} \mapsto e_{i_1} \otimes \cdots \otimes e_{i_r}.$$

Here, q-tensor space (an $S_q(n,r)$-module) is regarded as a U-module via the map θ. The element $e_{i_1} \otimes \cdots \otimes e_{i_r}$ is identified with the basis element $[A]$ where A has entries 1 in the (i_k, k)-positions and entries 0 elsewhere.

The first point to note is that the action of $u \in U$ on $[A]$ given by the map θ can easily be checked to be equivalent, under the identifications given, to the action of u on the corresponding element $e_{i_1} \otimes \cdots \otimes e_{i_r}$ via the comultiplication Δ. This is accomplished by checking the statement when u is one of the generators. Note that this is a special property of the comultiplication chosen.

The second important point is that any element of $T^r(M_n)$ given by $e_{i_1 1} \otimes \cdots \otimes e_{i_r r}$ is relaxed (and H-relaxed) and corresponds to one of the elements $[A]$ above. One sees now that in our action of $\xi \in S_v(n,r)$

on $V^{\otimes r}$, the coefficient of $e_{j_1} \otimes \cdots \otimes e_{j_r}$ in

$$\xi.(e_{i_1} \otimes \cdots \otimes e_{i_r})$$

is nothing other than the coefficient of $e_{j_1 i_1} \otimes \cdots \otimes e_{j_r i_r}$ in $\gamma(\xi)$.

The proof follows from these observations. ∎

4. On certain subalgebras of the q-Schur algebra

Using the description of the v-Schur algebra given in §3, we can describe a (fairly large) family of subalgebras of the q-Schur algebra which are spanned by certain natural basis elements e_A. To do this, we introduce the notion of a "good" set of matrix units.

Definition A subset G of $\{e_{ij} : 1 \leq i, j \leq n\}$ is called a *good* set of matrix units if it satisfies the following two conditions:

(i) G spans a subalgebra (possibly without a multiplicative identity) of the full matrix ring M_n;

(ii) if $e_{ab} \in G$ and $e_{cd} \in G$ where $a < c$ and $b > d$, then also $e_{ad} \in G$ and $e_{cb} \in G$.

Lemma 4.1 Let G be a good set of matrix units. Then $T^r(G)$ is a subalgebra of $T^r(M_n)$, and a right $\mathcal{H}$-submodule of $T^r(M_n)$.

Proof It is clear that $T^r(G)$ is a subalgebra of $T^r(M_n)$ because the elements of G span a subalgebra of M_n.

We now need to check that if $[e_{i_1 j_1} \otimes \cdots \otimes e_{i_r j_r}] \in T^r(G)$, then acting T_s (where $s = (p, p+1)$ for some p) on this element produces another element of $T^r(G)$. Because of the way the multiplication is defined, we need only check this in the case $r = 2$.

We now consider the matrix units appearing in the expression for $[e_{ab} \otimes e_{cd}].T_s$ (where $s = (12)$) as in the definition of the action of $\mathcal{H}$ on $T^r(M_n)$. It is easily checked that if $a \leq c$ and $b \leq d$, then no new matrix units can appear in the result. It is similarly easily checked that if $a > c$ and $b > d$, no new matrix units can appear in the result. If $a > c$ and $b = d$, or if $a = c$ and $b > d$, similar remarks hold.

We are left with the two most interesting cases, namely the case when $a > c$ and $b < d$, and the case when $a < c$ and $b > d$. In both cases, we find that e_{ad} and e_{cb} are also elements of G, from the definition of a good set of matrix units. It is now clear from the definition of the action of $\mathcal{H}$ that all matrix units appearing in the result are elements of G, as desired. ∎

Definition If G is a good set of matrix units, we denote by $S_v(G)$ the subspace of $S_v(n,r)$ spanned by all basis elements $[A]$ where A is a matrix such that $A_{ij} \neq 0 \Rightarrow e_{ij} \in G$. We denote by $S_q(G)$ the corresponding subspace of $S_q(n,r)$ spanned by elements e_A.

Lemma 4.2 Using the monomorphism γ defined in §3, the following identity holds in $T^r(M_n)$:

$$\gamma(S_v(n,r)) \cap T^r(G) = \gamma(S_v(G)).$$

Proof We first establish that $\gamma(S_v(G)) \leq \gamma(S_v(n,r)) \cap T^r(G)$. Clearly $S_v(G) \leq S_v(n,r)$. It is now enough to show that $\gamma([A]) \in T^r(G)$ where $[A]$ is a basis element of $S_v(G)$. By definition of γ, $\gamma([A])$ is of the form $t_A.h$ for some $h \in \mathcal{H}$. By the construction of $[A]$, we find that $t_A \in T^r(G)$, and therefore $\gamma([A]) \in T^r(G)$ by Lemma 4.1.

Now consider a typical element in $\gamma(S_v(n,r)) \cap T^r(G)$ given by

$$g = \sum_A c_A \gamma([A]),$$

where the c_A are elements of $\mathbf{Q}(v)$. We claim that unless $[A]$ is a basis element of $S_v(G)$, then c_A must be zero. This is achieved by considering relaxed basis elements occurring in the expression for g. These relaxed basis elements must all lie in $T^r(G)$. It follows from the definitions that the relaxed basis element corresponding to $[A]$ is an element of $T^r(G)$ if and only if $[A]$ is a basis element for $S_v(G)$. This establishes the reverse inclusion. ∎

Theorem 4.3 $S_v(G)$ is a subalgebra of $S_v(n,r)$ and $S_q(G)$ is a subalgebra of $S_q(n,r)$. The dimension of each subalgebra is

$$\binom{m+r-1}{r},$$

where m is the cardinality of the set G.

Proof We know that $S_v(G)$ is a subspace of $S_v(n,r)$ and it is clearly a subalgebra by Lemma 4.2, where it was shown to be an intersection of two subalgebras of $T^r(M_n)$. The corresponding result for $S_q(G)$ follows immediately by consideration of products $e_A e_{A'}$ where both elements are in $S_q(G)$.

By construction of the basis for $S_v(G)$ or $S_q(G)$, we see that the dimension is equal to the number of compositions of r into m pieces. The result now follows from elementary combinatorics. ∎

Examples

We now present some examples of good sets of matrix units.

1. Let

$$G = \{e_{ij} : 1 \leq i \leq j \leq n\}.$$

This spans the subalgebra of M_n given by upper triangular matrices. It is easily checked that G is good, and that $S_q(G)$ is the Borel subalgebra S^+.

2. Choose a finite sequence of integers r_i satisfying $0 = r_0 < r_1 < \ldots < r_s = n$. Then let

$$G = \{e_{ij} : 1 \leq i, j \leq n \ \wedge \ r_h < i \leq r_{h+1} \iff r_h < j \leq r_{h+1}\}.$$

This spans the subalgebra of M_n given by matrices of a particular block form (depending on the sequence r_i). The set G is good, and the algebra $S_q(G) \leq S_q(n, r)$ exists.

3. Let

$$G = \{e_{11}\} \cup \{e_{ij} : 2 \leq i \leq n, 1 \leq j \leq n\}.$$

This spans a subalgebra of M_n. Furthermore, G is good and the algebra $S_q(G) \leq S_q(n, r)$ exists.

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