MONOMIALS AND TEMPERLEY–LIEB ALGEBRAS

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Abstract. We classify the "fully tight" simply-laced Coxeter groups, that is, the ones whose iji-avoiding Kazhdan–Lusztig basis elements are monomials in the generators B_{s_i}. We then investigate the basis of the Temperley–Lieb algebra arising from the Kazhdan–Lusztig basis of the associated Hecke algebra, and prove that the basis coincides with the usual (monomial) basis.

1. Introduction

Let W be a Weyl group which we shall view as a Coxeter group with simple generators S. Every $w \in W$ may be written as a product $s_{i_1} \cdots s_{i_l}$ where $s_{i_k} \in S$. If l is minimal, we say this product is a reduced expression for w and we define the length of w to be $\ell(w) = l$.

Let $\mathcal{H}$ be the Hecke algebra associated with the Weyl group W. We understand this to be the $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$ algebra (where v is a square root of the indeterminate q associated with $\mathcal{H}$) with basis $\{T_w : w \in W\}$ and multiplication satisfying (1) $T_s^2 = (q-1)T_s + q$ for $s \in S$, and (2) $T_w T_{w'} = T_{ww'}$ whenever $\ell(ww') = \ell(w) + \ell(w')$. There is a unique automorphism $^- : \mathcal{H} \to \mathcal{H}$ which sends v to v^{-1} and T_w to $T_{w^{-1}}^{-1}$.

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According to Kazhdan and Lusztig [**9**], there is a unique basis B_w of $\mathcal{H}$ such that

$$B_w = \bar{B}_w = v^{-\ell(w)} \sum_{y \leq w} P_{y,w}(q) T_y,$$

where $\leq$ denotes the (strong) Bruhat order, and $P_{y,w}(q) \in \mathbb{Z}[q]$ is a polynomial of degree $\leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$ for $y < w$ and $P_{w,w}(q) = 1$. This basis is the basis denoted C'_w in [**9**].

Notice that for simple generators s, $B_s = v^{-1}(T_s + 1)$.

Let I be the two-sided ideal in $\mathcal{H}$ generated by elements of the form

$$\sum_{w \in P_2} T_w$$

where P_2 is a rank 2 proper parabolic of W not isomorphic to $A_1 \times A_1$. When the Hecke algebra is of type A, the quotient $\mathcal{H}/I$ is the Temperley–Lieb algebra which has been studied in, for instance, [**15**] and [**8**]. When the Dynkin diagram is simply laced, the quotient has been studied in [**3**] and [**4**], and the general cases have been tackled in [**5**] and [**14**].

The involution $^-$ induces an involution on the quotient $\mathcal{H}/I$, and $\mathcal{H}/I$ is equipped with a nice basis E_w. This basis is parametrized by the so-called "fully commutative" elements (as defined in [**14**, §0]). In the simply laced cases, this is the subset W_c of elements in W whose reduced expressions avoid substrings of the form iji where i and j are non-commuting simple generators. Such elements will be called *iji-avoiding*.

Because there are many properties of the E_w basis which mirror properties of the Kazhdan-Lusztig basis B_w, one might ask to what extent the two bases agree.

In particular, since the E_w are monomials in the E_s for $s \in S$, we ask, which B_w are monomials in the B_s for $s \in S$? This is the subject of section 2. Here we give a criterion for B_w to be a monomial in the B_s and determine explicitly which of the simply laced W have B_w consisting of the "maximal" number of monomials (see section 2 for a precise statement).

Also, we ask, does the basis B_w project to the basis E_w? This is the subject of section 3. In general, the ideal I is not compatible with the basis B_w. However in type A, it is, and here, E_w is indeed the projection of B_w.

Finally, we remark that if one is willing to use the positivity results concerning the structure constants of the basis B_w, some of our proofs can be simplified, for example, those in §3.7. However, since the positivity property is a deep result arising from the theory of perverse sheaves which would only help slightly in any case, we will prefer to use more elementary techniques instead.

2. Tight Monomials in Hecke Algebras

2.1 General results.

Motivated by Lusztig's paper [**12**], we say that B_w is a tight monomial and w is tight if B_w is a monomial in the B_s, $s \in S$. Denote by $T \subset W$ the set of $w \in W$ for which B_w is a tight monomial.

First, we give a criterion for determining whether B_w is a tight monomial.

Proposition 2.1.1. *Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression for w. Define $Q_y(q)$ by*

$$B_{s_{i_1}} \cdots B_{s_{i_l}} = \sum_{y \leq w} v^{-\ell(w)} Q_y(q) T_y.$$

Then B_w is tight if and only if $Q_y \in \mathbb{Z}[q]$ and $\deg Q_y < (\ell(w) - \ell(y))/2$ for $y \neq w$ and $Q_w = 1$.

Proof. Since $^-$ is an automorphism and $B_s = \bar{B}_s$ for $s \in S$, we see that $B_{s_{i_1}} \cdots B_{s_{i_l}}$ is fixed by $^-$. By the uniqueness properties of B_w, we can have $B_w = B_{s_{i_1}} \cdots B_{s_{i_l}}$ if and only if the $Q_y(q)$ are polynomials of a nature consistent with the nature of the $P_{y,w}(q)$. $\square$

We remark that this is equivalent to smallness of the standard desingularization map of the Schubert variety corresponding to w. (This follows easily from the relevant definitions.)

Lemma 2.1.2. *Let $w \in W$. Let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression for w. Assume that there exists $1 \leq k \leq n$ such that $s_{i_1} \cdots s_{i_{k-1}} s_{i_{k+1}} \cdots s_{i_n}$ (product of simple generators in the reduced expression in order, but omitting the kth term) is not reduced. Then w is not tight.*

Proof. Let $s_{j_1}, \ldots, s_{j_l}$ be any sequence of generators. Write

$$T_{s_{j_1}} T_{s_{j_2}} \cdots T_{s_{j_l}} = \sum R_y(q) T_y.$$

We claim that whenever $R_y \neq 0$, both the coefficient of $q^{\deg R_y}$ in R_y is positive and $\deg R_y \geq (l - \ell(y))/2$. We proceed by induction on l, the case where $l = 0$ or $l = 1$ being clear.

By definition, we have $T_s T_w = T_{sw}$ if $\ell(sw) > \ell(w)$ and $T_s T_w = (q-1)T_w + qT_{sw}$ if $\ell(sw) < \ell(w)$. For notational simplicity, we let $s = s_{j_1}$. Let $T_{s_{j_2}} \cdots T_{s_{j_l}} = \sum R'_z(q) T_z$. We have,

$$R_y(q) = \delta_{\ell(sy)+1,\ell(y)}(q-1)R'_y + \sum_{w,\ sw=y,\ \ell(w)<\ell(y)} R'_w + \sum_{w,\ sw=y,\ \ell(w)>\ell(y)} qR'_w.$$

By induction, we know that the coefficient of $q^{\deg R'_y}$ in R'_y is positive. Therefore, since the sum of polynomials in q which tend to infinity as q tends to infinity also tends to infinity with q, we see that the coefficient of $q^{\deg R_y}$ in R_y is positive as well. Also, since multiplication of an element in W by a simple generator changes its length by ± 1, we compute that $\deg R_y \geq (l - \ell(y))/2$.

Now consider the product expansion

$$v^{\ell(w)} B_{s_{i_1}} \cdots B_{s_{i_n}} = \sum_{y \leq w} Q_y(q) T_y.$$

This product is also a positive sum of monomials in the T_s where the monomials are indexed by subsequences of $s_{i_1} \cdots s_{i_n}$. Consider the monomial

$$T_{s_{i_1}} \cdots T_{s_{i_{k-1}}} T_{s_{i_{k+1}}} \cdots T_{s_{i_n}}.$$

After performing some braid relations (if necessary), we see that

$$T_{s_{i_1}} \cdots T_{s_{i_{k-1}}} T_{s_{i_{k+1}}} \cdots T_{s_{i_n}} = (q-1) T_{s_{j_1}} \cdots T_{s_{j_{n-2}}} + q T_{s_{k_1}} \cdots T_{s_{k_{n-3}}}.$$

Recall that we are assuming $s_{i_1} \cdots s_{i_{k-1}} s_{i_{k+1}} \cdots s_{i_n}$ is not reduced. By the claim, there exists y for which the coefficient of T_y in the expansion $T_{s_{j_1}} \cdots T_{s_{j_{n-2}}}$ is greater than or equal to $(\ell(w) - 2 - \ell(y))/2$. But this implies that there exists y such that $\deg Q_y(q) \geq 1 + (\ell(w) - 2 - \ell(y))/2 = (\ell(w) - \ell(y))/2$. (Recall that if $f(q)$ and $g(q)$ are polynomials which tend to infinity with q, then $\deg f + g = \max(\deg f,\ \deg g)$.)

By Proposition 2.1.1, this implies that w is not tight. $\square$

Proposition 2.1.3. *We have $T \subset W_c$, where W_c is the set of iji-avoiding elements.*

Proof. Suppose $w \notin W_c$. Then there exists a reduced expression $s_{i_1} \cdots iji \cdots s_{i_l}$ of w which involves a substring of the form iji where i and j are simple generators which do not commute. Consequently, the subsequence of length $l-1$ which omits j is not reduced. By lemma 2.1.2, w is not tight. $\square$

By the weak Bruhat order, we mean the order generated by the preorder on W defined by $w < ws$ if $\ell(ws) = \ell(w) + 1$ and $w < sw$ if $\ell(sw) = \ell(w) + 1$.

Proposition 2.1.4. *The set T is an order ideal in the weak Bruhat order on W.*

Proof. Consider the case where $w \in W$ and $s \in S$ are such that $\ell(sw) = \ell(w) + 1$. Suppose that $w \notin T$. Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression for w. Let

$$B_{s_{i_1}} \cdots B_{s_{i_l}} = \sum_{y \leq w} v^{-\ell(w)} Q_y(q) T_y.$$

Note that $Q_y(q)$ lies in $\mathbb{Z}[q]$; this can be seen by renormalising $B_{s_{i_1}}$ by multiplying by v and using the fact that the structure constants lie in $\mathbb{Z}[q]$. We therefore know from Proposition 2.1.1 that there exists y such that $\deg Q_y \geq (\ell(w) - \ell(y))/2$. By properties of the Hecke algebra multiplication, this implies that the coefficient of T_{sy} (respectively, T_y) has degree which violates if $\ell(sy) > \ell(y)$ (respectively, $\ell(sy) < \ell(y)$) the conclusion of proposition 2.1.1. Consequently sw is not tight. $\square$

2.2 Fully tight.

Since we know that T is an order ideal (with respect to weak Bruhat order) of W_c, and we know that W_c is also such an order ideal, it is natural to wonder whether $T = W_c$. We say W is fully tight if this is the case.

Proposition 2.2.1. *Among all simply laced Coxeter groups, A_1, A_2, A_3, A_4, A_5, A_6, D_4, and D_5 are the only fully tight ones.*

Proof. We used a computer to check that all the groups listed are fully tight.

To see that these are the only ones, we proceed as follows.

First note that if W is not fully tight, then no Coxeter group whose Coxeter graph contains a subgraph corresponding to the Coxeter graph of W is fully tight.

Let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression. By the expansion of w, we shall mean $T_{s_{i_1}} \cdots T_{s_{i_n}}$. (Note that this is independent of reduced expression.)

In type A_7, we label the generators 1 through 7 in the standard way. Consider $w = 4567\ 3456\ 2345\ 1234$. This is the longest element in W_c (see [**3**]). Consider the product $X = T_4T_5\ T_7\ T_3T_4T_5\ T_3T_4T_5\ T_1\ T_3T_4$ formed by a subsequence of the given reduced expression for w. Note that 345 345 is the longest element of an A_3 parabolic subgroup of A_7 so that 3, 4, and 5 shorten it when multiplied on either side. One sees that the coefficient of $T_{7\ 345\ 345\ 1}$ in the expansion of X is therefore of degree at least $4 \geq (16-8)/2$. (In fact the degree is equal to 4.) Therefore w is not tight.

In D_6, we label the generators 1, $\bar{1}$, and 2 through 5 in the standard way (so 1 and $\bar{1}$ commute with each other, but not with 2). A similar argument to the case for A_7 shows that $w = 1\ 2\bar{1}\ 321\ 432\bar{1}\ 54321$ is not tight. (This w is one of the two longest elements in W_c.) The coefficient of $T_{1\ 21\ \bar{1}\ 34321}$ has degree $3 \geq (15-9)/2$.

In E_6, we label the generators 0 through 5 so that 1 through 5 generate an A_5 parabolic subgroup. A similar argument to the case for A_7 and D_6 shows that $w = 5430213243054321$ is not tight. (This w is again, one of the two longest elements in W_c.) The coefficient of $T_{0345430213}$ has degree $3 \geq (16-10)/2$.

In $\hat{D}_4$, we label the generators m, 1, 2, 3, and 4, where 1, 2, 3, and 4 all commute with each other, but none of which commute with m. Let $w = 1m23m41m23m1$. Note that $w \in W_c$. However,

$$\begin{aligned} 1m23m1m23m1 &= 1m123m321m1 \\ &= m1m2m3m2m1m \\ &= m12m232m21m \\ &= m12m3m21m. \end{aligned}$$

Therefore, by lemma 2.1.2, w is not tight.

Finally, in $\hat{A}_{n-1}$, for $n > 2$, we label the generators 1 through n in the standard way. Let $v = 1234 \cdots (n-1)$. Let $w = vnv$. Then $w \in W_c$ (in fact, w has a unique reduced expression). However, we claim that vv is not reduced. To see this, note that $v^{-1}v^{-1}(-\alpha_1) = -\alpha_{n-1}$ where α_1 and α_{n-1} are the roots which define the simple reflections 1 and $n-1$. Therefore, by lemma 2.1.2, w is not tight.

To summarize, we have shown that if W is fully tight with simply laced Coxeter graph Γ, then

1. Γ contains no loops,
2. Γ contains no string of length 7,
3. Γ contains no node with more than 3 branches,
4. Γ does not contain a subgraph of type D_6 or E_6.

These eliminate all simply laced graphs not listed in the statement of proposition 2.2.1. $\square$

3. The Main Results

3.1 The Kazhdan–Lusztig basis for the Temperley–Lieb algebra.

From now on we consider only the Temperley–Lieb algebra TL_r corresponding to a Dynkin diagram of type A_{r-1}. This is given by generators $E_1, \ldots, E_{r-1}$ and defining relations

$$\begin{aligned} E_i^2 &= [2]E_i, \\ E_iE_j &= E_jE_i, \quad \text{if } |i-j|>1, \\ E_iE_{i\pm1}E_i &= E_i. \end{aligned}$$

Here, the symbol $[2]$ denotes the Laurent polynomial $v + v^{-1}$.

Using the basis $\{B_w : w \in W\}$ of the Hecke algebra $\mathcal{H}$ in §2, we can obtain a basis $\{F_w : w \in W_c \subset W\}$ of the Temperley–Lieb algebra $\mathcal{H}/I$. In general, I is not spanned by the basis elements B_w it contains. However, in type A we have the following result.

Proposition 3.1.1. *The kernel of $\sigma : \mathcal{H}(A_{r-1}) \to \mathcal{H}(A_{r-1})/I$ is spanned by the elements*

$$\{B_w : w \notin W_c\},$$

where W_c is as defined in section 1. In fact, W_c is a union of two-sided Kazhdan–Lusztig cells in type A.

Proof. The first assertion follows from the second. The second part follows from [**3**, Proposition 6] and the fact that the two-sided Kazhdan–Lusztig cells in type A agree with the Robinson–Schensted correspondence. (This latter fact can be deduced from results in [**10**, Theorem 3.8], and details of the Robinson–Schensted correspondence may be found in [**13**].) Here, W_c corresponds to pairs of tableaux with strictly fewer than 3 columns, and all such pairs of tableaux turn up in this way. □

Definition 3.1.2. *The elements* $\{F_w : w \in W_c\}$ *are given by* $F_w := \sigma(B_w)$.

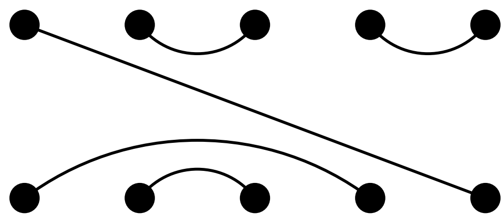
It is clear that the elements $\{F_w : w \in W_c\}$ form a basis for TL_r.

3.2 The r-diagram calculus.

We now recall the r-diagram calculus for the Temperley–Lieb algebra. Full details may be found in [**16**, §1].

An r-diagram consists of two rows of r nodes together with r edges linking pairs of nodes. Each node is the endpoint of exactly one edge, and the edges must not intersect. An example of an r-diagram for $r = 5$ is given in Figure 1.

FIGURE 1: AN r-DIAGRAM ($r = 5$)

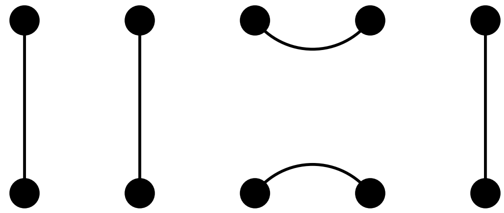


The r-diagrams can be thought of as basis elements for the Temperley–Lieb algebra. The multiplication is given in a natural way as follows. Suppose A and B are r-diagrams. Place the A above B, identifying the lower row of A with the upper row of B, and then remove all the nodes in this new combined row. This will produce x closed loops, where x is possibly zero. Remove these closed loops to form a new r-diagram, C. Then the product AB in the algebra is given by $[2]^x C$.

We call edges joining points in the same row *horizontal*, and edges joining points in different rows *vertical*. It is clear that the number of horizontal edges in the top row equals the number of horizontal edges in the bottom row.

The generator E_i of the algebra TL_r is identified with the diagram in which the i and $i + 1$ positions in both rows are joined by horizontal edges, and the other points are joined by vertical edges. Figure 2 shows the example of E_3 in the case $r = 5$.

FIGURE 2: THE GENERATOR E_3 FOR $r = 5$



We now show that the basis of r-diagrams may naturally be parametrised by the elements of W_c as follows.

Definition 3.2.1. *For each $w \in W_c$, we define $E_w \in TL_r$ to be $E_{i_1} \cdots E_{i_l}$, where $s_{i_1} \cdots s_{i_l}$ is a reduced expression for w.*

Notice that since $w \in W_c$, this definition doesn't depend on the reduced expression taken for $w \in W_c$, because in this case any reduced expression for w can be obtained from any other by repeated application of the relation $s_i s_j = s_j s_i$ (for $|i - j| > 1$), and in this case, $E_i E_j = E_j E_i$ also holds.

Proposition 3.2.2. *The set $\{E_w : w \in W_c\}$ coincides with the basis of r-diagrams.*

Proof. It was shown in [**3**, §2] that the elements $\{\tau_w : w \in W_c\}$ form a free $\mathcal{A}$-basis for TL_r, where $\tau_w = v^{\ell(w)} E_w$. So clearly the set $\{E_w : w \in W_c\}$ forms a free $\mathcal{A}$-basis.

By considering the representation of the algebra generators E_s as r-diagrams, we see that each E_w is equal to a power of [2] multiple of some r-diagram. Since [2] is not a unit and both the set of r-diagrams and the set $\{E_w : w \in W_c\}$ are free $\mathcal{A}$-bases for TL_r, we deduce that all the powers of [2] occurring are trivial, and that the two bases coincide. □

From now on, we will refer to the basis E_w (i.e. the basis of r-diagrams) as the *monomial* basis. The main aim of §3 is to prove that the basis F_w coincides with the monomial basis.

3.3 Two-sided cells.

An important feature of the Temperley–Lieb algebras which is emphasised by properties of either the E_w or the F_w basis is that they are cellular algebras (in the sense of [**6**]).

The two-sided cells with respect to the basis $\{F_w : w \in W_c\}$ are defined via the Robinson–Schensted correspondence. Under this correspondence, the elements of W_c correspond to pairs of standard tableaux of the same shape with at most two columns. The shapes of these tableaux are clearly determined by the length of the second column, which is an integer between 0 and $\lfloor \frac{r}{2} \rfloor$ inclusive. The term "cell" on its own should be understood to mean a two-sided cell.

The two-sided cells for Hecke algebras of symmetric groups, such as F-cells we have been studying, are well known to be the disjoint union of *left cells* and also the disjoint union of *right cells*, as one finds from the Robinson–Schensted correspondence. (For the definitions, see [**7**, §7.15].) All the left and right cells are the same size as each other, and any given left cell and right cell in the same two-sided cell intersect in a unique element. We use the usual notation $w_1 \sim_L w_2$, $w_1 \sim_R w_2$, and $w_1 \sim_{LR} w_2$ to mean that two elements of the group (or their associated basis elements in the algebra) w_1 and w_2 are in the same left cell, right cell, and two-sided cell, respectively.

We will use the fact (due to Lusztig, and mentioned for example in [**1**, §1.3]) that if $x \leq_L y$ then there exists $h \in \mathcal{H}$ such that B_x appears in the expansion of the product hB_y with respect to the B-basis. Here, $\leq_L$ is a certain natural order on the elements of a Coxeter group, and the property we have just cited may be regarded as the definition of the order.

Definition 3.3.1. *For each integer k between 0 and $\lfloor \frac{r}{2} \rfloor$, we define the k-th F-cell to be the set of F_w where w lies in the two-sided Kazhdan–Lusztig cell corresponding to pairs of tableaux with at most two columns and exactly k boxes in the second column. One sees easily from the Robinson–Schensted correspondence that the elements $w \in W$ arising from this procedure lie in W_c, and that each element of W_c turns up in this way for a suitable k.*

We will say that the i-th F-cell is greater than the j-th F-cell if and only if $i > j$.

The two-sided cells with respect to the basis $\{E_w : w \in W_c\}$ may be understood in terms of the *parenthesis diagrams* of [**16**, §2].

Definition 3.3.2. *An (r,p)-parenthesis diagram consists of a row of r points (imagined to be on the x-axis of the plane) and a set of p edges, each of which lies in the lower half of the plane, joins exactly two of the points, and does not intersect any of the other edges. If points a,b,c satisfy $a < b < c$ and a is connected to c, then b is required to be connected to some other point.*

The motivation behind this definition is that any r-diagram, A, now corresponds naturally to a pair of (r,p)-parenthesis diagrams, where p is half the number of horizontal edges in A. This can be seen by removing all the vertical edges from A: the upper half of the diagram is an (r,p)-parenthesis diagram (call it a_1), and the lower half is an inverted (r,p)-parenthesis diagram (call this a_2). Note that the diagram A can be recovered from the ordered pair (a_1, a_2) corresponding to it because of the requirement that the edges of A do not intersect (meaning that there is a unique way of connecting the endpoints of vertical edges).

Following Westbury's use [**16**, §5] of the Dirac "dyadic" notation, we denote the diagram with a_1 on top and a_2 on the bottom by $|a_1\rangle\langle a_2|$.

Definition 3.3.3. *For each integer k between 0 and $\lfloor \frac{r}{2} \rfloor$ we define the k-th E-cell to be the set of basis elements $|a\rangle\langle b|$ where a and b are (r,k)-parenthesis diagrams. We order the E-cells in the obvious way based on the values of k.*

We write $a \sim b$ to mean that a and b are (r,k)-parenthesis diagrams for the same values of r and k.

We say that $|x\rangle\langle y|$ and $|a\rangle\langle b|$ are in the same left E-cell if and only if $y = b$, and we say they are in the same right E-cell if and only if $x = a$.

Definition 3.3.4. *We denote by L_k the number of standard tableaux having at most two columns with $r-k$ entries in the first column and k entries in the second column.*

Lemma 3.3.5. *The size of the k-th E-cell and the size of the k-th F-cell are both given by L_k^2.*

Proof. Westbury explains in [**16**, §2] how to associate to each (r,p)-parenthesis diagram a pair of standard tableaux with r boxes, at most two rows, and p entries in each of the second rows. Clearly the same holds for the transposes of these tableaux (i.e. those with at most two columns). This means that the size of the k-th E-cell is L_k^2. One sees from the Robinson–Schensted correspondence that this is also the size of the k-th F-cell. $\square$

3.4 Ideals and modules over a field.

In this section, we change the base ring of the Temperley–Lieb algebra from $\mathcal{A}$ to $\mathbb{Q}(v)$, and study $\mathbb{Q}(v) \otimes TL_r$, where $\mathbb{Q}(v)$ is the field of fractions of $\mathcal{A}$. We then study certain two-sided ideals and modules in TL_r over the field.

Definition 3.4.1. *For each integer k indexing one of the cells (with respect to either basis), we define the (r,k)-parenthesis diagram $\ell(k)$ to be that in which the nodes labelled $2i+1$ and $2i+2$ are joined by an edge if and only if $i < k$. We denote the E-basis element $|\ell(k)\rangle\langle\ell(k)|$ by $D(k)$. It is also equal to F_{w_k} where $w_k := s_1 s_3 \cdots s_{2k-1}$ (meaning w_k is the identity when $k = 0$); this follows from the definition of the F-basis because each s_i occurring in the expression for w_k occurs at most once.*

Using the elements $\ell(i)$ from above, we now define a set $\{I_k\}$ of two-sided ideals of TL_r (over $\mathbb{Q}(v)$) as follows.

Definition 3.4.2. *Let k be the label of one of the cells of TL_r. The ideal I_k of $\mathbb{Q}(v) \otimes TL_r$ is defined to be*

$$\langle D(k') : k' \geq k \rangle.$$

Next, we use the ideals introduced above to define a certain left module for TL_r over the field.

Definition 3.4.3. *Let k be the label of one of the cells. We denote by M_k the TL_r-submodule of TL_r/I_{k+1} generated by $v_k := D(k) + I_{k+1}$. We understand I_{k+1} to mean 0 if $k+1$ is not the label of one of the cells.*

Lemma 3.4.4. *The dimension of I_k is given by $\sum_{k' \geq k} L_{k'}^2$, where L_k is the number of standard tableaux with at most two columns and k boxes in the second column.*

The ideal I_k is spanned by the F_w lying in the k-th and higher F-cells, and also by the E_w lying in the k-th and higher E-cells.

Proof. One may easily check from the definitions for the E-cells and the F-cells that the elements $D(k')$ (where $k' \geq k$) are elements of both the E-basis and the F-basis. Moreover the element $D(k')$ lies in the k'-th E-cell and the k'-th F-cell. It follows from [**6**, §1.2] that the F-basis is a cellular basis, and it was shown in [**6**, Theorem 6.7] that the E-basis is a cellular basis. This means that the ideal generated by the $D(k')$ (for $k' \geq k$) is contained in the span of the union of the cells as stated in the statement of the Lemma. Hence it is enough to prove that any E_w lying in the k-th or higher E-cells is in the ideal I_k, because then the dimension will be as asserted (by Lemma 3.3.5), and the statement for the F-basis will follow, also by Lemma 3.3.5.

To prove this assertion, let $|a\rangle\langle b|$ be a typical element of the k-th E-cell. The proof follows from the observation that

$$[2]^{2k}|a\rangle\langle b| = |a\rangle\langle \ell(k)| \times |\ell(k)\rangle\langle \ell(k)| \times |\ell(k)\rangle\langle b|.$$

Since we are working over $\mathbb{Q}(v)$, $[2]^{2k}$ is invertible, and the proof follows. □

3.5 The a-function.

We now recall the $\mathbf{a}$-function on the Weyl group W. This was studied extensively in the series of papers [**11**], but there using a basis for the Hecke algebra $\mathcal{H}(W)$ which differs somewhat from ours. Our approach is more similar to that followed in [**2**], which uses the B_w basis of $\mathcal{H}$.

Definition 3.5.1. *Let $g_{x,y,z}$ be one of the structure constants for the basis $\{B_w : w \in W\}$ of $\mathcal{H}(W)$, namely*

$$B_x B_y = \sum_z g_{x,y,z} B_z.$$

Define, for $z \in W$,

$$\mathbf{a}(z) = \max_{x,y \in W} \deg(g_{x,y,z}).$$

The following properties of the **a**-function were proved in [**11**] and turn out to have important consequences for the F-basis:

Proposition 3.5.2.

(a) *The* **a***-function is constant on two-sided cells.*

(b) *For all* $z \in W$, $\mathbf{a}(z) \leq \ell(z)$.

(c) *The bound* $\max_{x,y \in W} \deg(g_{x,y,z})$ *can only be achieved when* $x \sim_L y^{-1}$, $y \sim_L z$, $x \sim_R z$ *and* $\mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(z)$.

(d) *There is a unique involution* d *such that* $d \sim_R x$. *Then* $g(d, x, x)$ *has largest possible degree, and no other involution* $d' \sim_{LR} x$ *has this property. Furthermore,* $g(d, x, x)$ *has leading coefficient 1.*

(e) *There is a unique involution* d *such that* $d \sim_L x$. *Then* $g(x, d, x)$ *has largest possible degree, and no other involution* $d' \sim_{LR} x$ *has this property. Furthermore,* $g(x, d, x)$ *has leading coefficient 1.*

(f) *For each* x *there is a unique involution* d *such that* $g(x, x^{-1}, d)$ *has maximum degree and leading coefficient* 1.

Proof. Proofs of (a) and (c) appear in [**11**, I], and proofs of the other statements appear in [**11**, II]. □

Since the homomorphism σ respects the two-sided cells, we can define an **a**-function for the F-basis in a natural way. This leads to the following result:

Lemma 3.5.3. *Consider the two-sided cell of* $\mathcal{H}$ *corresponding to the* k*-th* F*-cell. The* **a***-function takes value* k *on this cell.*

Proof. The k-th F-cell contains the element w_k, where w_k is as in Definition 3.4.1. By Proposition 3.5.2 (b), $\mathbf{a}(w_k) \leq \ell(w_k) = k$. Conversely, $B_{w_k}^2 = [2]^k B_{w_k}$, and $[2]^k$ has degree k. So $\mathbf{a}(w_k) \geq k$, forcing $\mathbf{a}(w_k) = k$. The assertion follows from 3.5.2 (a). □

Definition 3.5.4. *An element* E_w *is called involutory if and only if it is of the form* $|a\rangle\langle a|$. *It is clear that there is a unique involutory element in each left* E*-cell and in each right* E*-cell. If* E_w *is involutory, we denote it by* $E_{d,d}$, *where the* *symbols d are in bijection with the left cells. We write $E_{d,d'}$ for the element of the E-basis in the same right cell as the involutory element corresponding to d and the same left cell as the involutory element corresponding to d'. Any element E_x may thus be written in the form $E_{d,d'}$.*

This **a**-function can also be applied to the E-basis of TL_r, where we have a result analogous to Proposition 3.5.2:

Lemma 3.5.5.

(a) *Let E_x, E_y, E_z be elements of the E-basis in the cells numbered k_x, k_y, k_z respectively. Write*

$$E_x E_y = \sum_{z'} g_{x,y,z'} E_{z'}.$$

Then $\deg(g_{x,y,z}) \leq k_z$, and the bound can be achieved only if $k_x = k_y = k_z$.

(b) *Let $E_x = E_{d,d'}$ lie in the k-th E-cell. Then $E_{d,d}$ is the unique involutory element in the k-th E-cell for which the product $E_{d,d}E_{d,d'}$ produces a Laurent polynomial of maximum allowable degree. Similarly, $E_{d',d'}$ is the unique involutory element in the k-th E-cell such that $E_{d,d'}E_{d',d'}$ produces a Laurent polynomial of maximum allowable degree.*

Proof. The degree of a resulting polynomial g resulting from a multiplication is the number of loops formed by concatenation of the relevant diagrams. The r-diagram of an element E_w from cell number k has two sets of k horizontal edges, so can form at most k loops when multiplied by another element. The bound can only be achieved for products $|a_1\rangle\langle b_1||a_2\rangle\langle b_2|$ where $b_1 = a_2$.

The proof follows easily from these observations. □

3.6 Agreement of the bases on left cells, up to sign.

In this section, we continue to work over $\mathbb{Q}(v)$, unless otherwise stated.

Lemma 3.6.1.

(a) *There is a unique $\mathbb{Q}$-linear anti-automorphism $*$ of TL_r sending v to v^{-1} and F_s to F_s.*

(b) *For all $w \in W_c$, $F_w^* = F_{w^{-1}}$. The image of $|a\rangle\langle b|$ under $*$ is $|b\rangle\langle a|$.*

Proof. There is an anti-automorphism j of $\mathcal{H}$ which sends T_y to T_y^{-1}, and hence sends B_w to B_w^{-1}. Composing this map with the map $^-$ sending v to v^{-1} induces a map $*$ on $\mathcal{H}/I$ (as required in (a)), because I is stable under $^-$ and j. Uniqueness follows from the fact that the F_s are algebra generators.

The first part of (b) follows similarly. For the second part, it is clear [see **16**, §5] that sending $|a\rangle\langle b|$ to $|b\rangle\langle a|$ and v to v^{-1} is an anti-automorphism, and it is the same as $*$ because they agree on the base ring and on the generators F_s. This completes the proof of (b). $\square$

Lemma 3.6.2. *The F-basis and the E-basis induce the same basis on the module M_k up to sign. That is, the subset of M_k given by*

$$\{F_w + I_{k+1} : w \sim_L w_k\}$$

coincides with the subset of M_k given by

$$\{\pm|a\rangle\langle\ell(k)| + I_{k+1} : a \sim \ell(k)\},$$

for a suitable choice of signs. (Recall that $\ell(k)$ appeared in Definition 3.4.1.)

Proof. Consider a typical F_w satisfying $w \sim_L w_k$. Then there exists $f \in TL_r$ such that F_w appears with nonzero coefficient in the expansion of $f \cdot F_{w_k}$ with respect to the F-basis. Clearly we can choose f to be an element of the E-basis, i.e. a monomial $F_{i_s}F_{i_{s-1}} \cdots F_{i_1}$, where we write F_{i_t} for $F_{s_{i_t}}$ for typographical convenience. Let us pick such an f where s is as small as possible and consider the product

$$F_{i_s} \cdots F_{i_1} F_{w_k} = fF_{w_k}.$$

It is enough to prove by induction on p that

$$F_{i_p}F_{i_{p-1}} \cdots F_{i_1}F_{w_k} = \pm F_y \mod I_{k+1} = \pm E_x \mod I_{k+1},$$

for a certain $y \sim_L w_k$, and a certain element of the E-basis which we denote by $E_{x'}$.

The case $p = 0$ is obvious. We may now assume that the statement is true for p and prove that $F_{i_{p+1}} F_y = \pm F_{x'} \mod I_{k+1} = \pm E_{x'} \mod I_{k+1}$, where

$$F_y = \pm F_{i_p} \cdots F_{i_1} F_{w_k}.$$

Since $F_w \notin I_{k+1}$, the same is true for F_y and $F_{i_{p+1}} F_y$. Consideration of the multiplication properties of the E-basis shows that $E_{i_{p+1}} E_x$ can only equal $[2] E_x$ modulo I_{k+1} or $E_{x'}$ modulo I_{k+1} for some suitable x'. The former case is impossible as it would imply $F_{i_{p+1}} F_{x'} = [2] F_{x'}$ which is not the case, by minimality of s. So we have shown that

$$F_{i_{p+1}} F_{x'} = \pm E_{x'} \mod I_{k+1}.$$

Since $F_{i_{p+1}} F_{x'} \neq [2] F_{x'} \mod I_{k+1}$, it follows from [**7**, §7.14] that

$$F_{i_{p+1}} F_{x'} = F_{z_0} + \sum_z \mu(z, x) F_z,$$

where $z_0 = s_{i_{p+1}} x' > x$ and the scalars $\mu(z, x)$ are integers. Using the fact [**11**, I, Corollary 6.3 (c)] that $x \leq_L y$ and $x \sim_{LR} y$ imply $x \sim_L y$, we have

$$F_{i_{p+1}} F_{x'} = \sum_{z \sim_L w_k} \mu(z, x) F_z \mod I_{k+1}, \tag{1}$$

where we define $\mu(z_0, x) = 1$ if $z_0 \sim_L w_k$ and 0 otherwise, for notational convenience.

We now consider the coefficient of $F_{w_k} = |\ell(k)\rangle\langle\ell(k)|$ in

$$v^{-k} (F_{i_{p+1}} F_{x'})^* v^{-k} (F_{i_{p+1}} F_{x'}).$$

We know this is equal to $1 \mod v^{-1}\mathbb{Z}[v^{-1}]$ because it is clear that $E^*_{x'} E_{x'} = [2]^k F_{w_k}$. If $z, z' \sim_L w_k$ then it follows from Proposition 3.5.2, parts (c) and (f), that the coefficient of F_{w_k} in $v^{-k}(F_z)^* v^{-k} F_{z'}$ is $0 \mod v^{-1}\mathbb{Z}[v^{-1}]$ unless $z = z'$ in which case the coefficient is $1 \mod v^{-1}\mathbb{Z}[v^{-1}]$. Using these observations and the equation (1), we find that only one F_z can appear in the expansion of (1) with nonzero coefficient, and that it appears with coefficient ± 1. $\square$

3.7 Agreement of the bases on the two-sided cells, up to sign.

We continue to work over $\mathbb{Q}(v)$, and consider the ${}_{\mathcal{A}}TL_r$-bimodule

$$T_k = (I_k \cap {}_{\mathcal{A}}TL_r) + I_{k+1}.$$

Lemma 3.7.1. *The subset of T_k given by*

$$\{F_w + I_{k+1} : w \sim_{LR} w_k\}$$

coincides with the subset of T_k given by

$$\{\pm|a\rangle\langle b| + I_{k+1} : a \sim \ell(k) \sim b\},$$

for a suitable choice of signs.

Proof. It follows from [**6**, §1.2] that the F-basis is cellular, and it was shown in [**6**, Theorem 6.7] that the E-basis is cellular. The theory of cellular algebras (see [**6**, (2.4.1)]) now shows that T_k is a disjoint union of left submodules, each of which is spanned by the images of the basis elements which it contains, with respect to either basis. It is also a disjoint union of right submodules with similar properties. The left (respectively right) submodules are called left (respectively right) cells. Any given left cell and any right cell arising from T_k (which corresponds to a two-sided cell) intersect in a 1-dimensional space spanned by one of the basis elements.

Since the anti-automorphism $*$ agrees with the cellular anti-automorphism $*$ in [**16**, §5] or [**6**, Theorem 6.7] (provided we take the parameter to be [2] and not v), one of the right cells with respect to the F-basis can be easily seen to be

$$\{F_w + I_{k+1} : w \sim_R w_k\},$$

which is also (by applying $*$ to Lemma 3.6.2) one of the right cells with respect to the E-basis, up to sign, namely

$$\{\pm|\ell(k)\rangle\langle x| + I_{k+1}\}.$$

Now consider an arbitrary element $|a\rangle\langle b|$ in the two-sided cell labelled by k. This is of form

$$\frac{1}{[2]^k}|a\rangle\langle\ell(k)||\ell(k)\rangle\langle b| + I_{k+1},$$

and hence, by Lemma 3.6.2, of form

$$\pm \frac{1}{[2]^k} F_{w_1} F_{w_2} + I_{k+1},$$

where $F_{w_1} + I_{k+1} = \pm|a\rangle\langle\ell(k)| + I_{k+1}$ and $F_{w_2} + I_{k+1} = \pm|\ell(k)\rangle\langle b| + I_{k+1}$. Since the F-basis is cellular, we must have

$$\frac{1}{[2]^k} F_{w_1} F_{w_2} + I_{k+1} = cF_w + I_{k+1},$$

for some $w \sim_{LR} w_k$ and a scalar c. Because the F-basis and the E-basis give the same $\mathcal{A}$-form, $c \in \mathcal{A}$.

Applying the anti-automorphism $*$ to the above equation shows that c is symmetric in v and v^{-1}. Regarding T_k as an algebra in the obvious way, and M_k (considered over $\mathcal{A}$) as a left ideal of T_k, we find that $(M_k M_k^*)/[2]^k$ is equal to T_k; this is clear with respect to the E-basis. Therefore c must be a unit, and the other conditions force $c = \pm 1$. This completes the proof. $\square$

Now we have a canonical correspondence between the elements of the F-cells and the elements of the E-cells. It will be convenient for the purposes of the next section to introduce a parallel notation for these two bases in terms of certain idempotents.

It follows from standard properties of the Robinson–Schensted correspondence that each F-cell c has a subset of elements $\mathcal{D}_c$ which correspond to involutions in the Weyl group. There is exactly one element of $\mathcal{D}_c$ in each left cell, and exactly one in each right cell. We can therefore denote F_w by $F_{d,d'}$, where d is the involution in the same right cell as w, and d' is the involution in the same left cell as w. For $d \in \mathcal{D}_c$, $F_d = F_{d,d}$, and $F_d^* = F_d$. The element E_w corresponding to F_d in the sense of Lemma 3.7.1 also clearly satisfies $E_w^* = E_w$, and is therefore involutory. We may thus take the symbols d in Definition 3.5.4 to be the iji-avoiding involutions.

3.8 Agreement of the bases.

Lemma 3.7.1, together with the fact that the $\mathbf{a}$-function takes value k on the k-th cell (for either basis) motivates the following definition:

Definition 3.8.1. *We denote by $\mathcal{L}_k$ the $\mathbb{Z}[v^{-1}]$-lattice with basis*

$$\{v^{-k} F_w + I_{k+1} : w \sim_{LR} w_k\},$$

or (equivalently by Lemma 3.7.1) with basis

$$\{v^{-k}|a\rangle\langle b| + I_{k+1} : a \sim \ell(k) \sim b\},$$

which is the same basis up to sign.

We are now ready to prove the main result of this section.

Theorem 3.8.2. *The F-basis and the E-basis are identical; that is, if $s_{i_1} \cdots s_{i_m}$ is a reduced expression for $w \in W_c$, then*

$$F_w = F_{i_1} \cdots F_{i_m},$$

where we write F_i for F_{s_i}.

Proof. We proceed by induction on $n = \ell(w)$, the cases of $n = 0$ and $n = 1$ being trivial.

Otherwise we can write $w = sw'$, where s is a simple reflection and $\ell(w') = \ell(w) - 1$. It is clear that $w' \in W_c$. From [**7**, §7.14] we see that

$$F_s F_{w'} = F_{sw'} + \sum_z \mu(z, w') F_z,$$

where the $\mu(z, w')$ are certain integers arising from the theory of Kazhdan–Lusztig polynomials. (Note that in [**16**, §7], something similar is considered, but the function μ occurring there is defined in an ad hoc way, not from the Kazhdan–Lusztig theory.) It is important for the proof that the structure constants occurring are integers; this will be used implicitly from now on.

We now argue that in fact $F_s F_{w'} = F_{sw'} = F_w$.

We wish to find the coefficient of $F_{d,d'}$ (a basis element in the k-th cell) in the product $F_s F_{w'}$. To do this, we consider the coefficient c of $F_{e,e'}$, another typical element in the k-th cell, in the product

$$\Pi = v^{-\mathbf{a}(k)} F_{d,d} \left(v^{-\mathbf{a}(k)} F_w + \sum_z \mu(z, w) v^{-\mathbf{a}(k)} F_z \right) v^{-\mathbf{a}(k)} F_{d',d'}.$$

It follows from the definition of the $\mathbf{a}$-function that c is an element of $v^{-\mathbf{a}(k)}\mathbb{Z}[v^{-1}]$; in other words, after quotienting by I_{k+1}, this expression lies in the lattice $\mathcal{L}_k$. Let c' be the coefficient of $v^{-\mathbf{a}(k)}$ in the polynomial c, i.e. the coefficient of $v^{-\mathbf{a}(k)}F_{d,d'}$ after projection to the quotient lattice $\mathcal{L}_k/v^{-1}\mathcal{L}_k$.

By Proposition 3.5.2, part (c) and Lemma 3.5.3, the only F_z appearing with nonzero coefficient in $F_sF_{w'}$ which can contribute to c' are those in the k-th F-cell. By Proposition 3.5.2, parts (d) and (e), the only F_z which can contribute is in fact $F_{d,d'}$ itself, and furthermore, c' is the coefficient with which $F_{d,d'}$ occurs in $F_sF_{w'}$.

In the context of the quotient lattice $\mathcal{L}_k/v^{-1}\mathcal{L}_k$, this means that the image of Π in the quotient lattice is well defined, and the coefficient of the basis element of $\mathcal{L}_k/v^{-1}\mathcal{L}_k$ corresponding to $F_{d,d'}$ is the coefficient of $F_{d,d'}$ in $F_sF_{w'}$.

By Lemma 3.7.1, $F_{d,d} = \pm E_{d,d} \mod I_{k+1}$ and $F_{d',d'} = \pm E_{d',d'} \mod I_{k+1}$, so it is also true that the coefficient of $F_{d,d'}$ in $F_sF_{w'}$ is the (integer) coefficient of $v^{-\mathbf{a}(k)}$ in the (polynomial) coefficient of $F_{d,d'}$ in

$$\pm v^{-\mathbf{a}(k)}E_{d,d}\left(v^{-\mathbf{a}(k)}F_w + \sum_z \mu(z,w)v^{-\mathbf{a}(k)}F_z\right)v^{-\mathbf{a}(k)}E_{d',d'}.$$

Because F_w is not a scalar multiple of $F_sF_{w'}$, the induction hypothesis shows that $F_sF_{w'}$ is a member of the E-basis (as opposed to being [2] times a member of the E-basis), which we denote by E_x.

We now define the product

$$\Pi' = v^{-\mathbf{a}(k)}E_{d,d}\left(v^{-\mathbf{a}(k)}E_x\right)v^{-\mathbf{a}(k)}E_{d',d'}$$

We consider the coefficient of $E_{e,e'}$ in this expression, where $E_{e,e'}$ is a typical element in the same E-cell as $E_{d,d'}$. Using similar techniques to the above, but using Lemma 3.5.5 instead of Proposition 3.5.2, we find that again it makes sense to take the image of Π' in the quotient lattice, and that the coefficient of $E_{e,e'}$ in Π' is the same as the coefficient of the basis element in the quotient lattice which corresponds to $E_{e,e'}$.

It was remarked in Definition 3.8.1 that the E-basis and F-basis for the quotient lattice are the same up to sign. Since we know only one E-basis element occurs in the product $F_sF_{w'}$, it follows that only one F-basis element can occur in the product $F_s F_{w'}$. But certainly F_w occurs, and with coefficient 1. Thus $F_s F_{w'} = F_w$ as required, and the theorem is proved. $\square$

A natural question to ask is whether Theorem 3.8.2 generalises to the Hecke algebra quotients $\mathcal{H}/I$ for other types of Dynkin diagrams. Unfortunately, Proposition 3.1.1 is in general false: it even fails for type D_4, where the iji-avoiding elements are not a union of two-sided Kazhdan–Lusztig cells. This means that there is no obvious generalisation of Theorem 3.8.2 to the other simply-laced cases. It is possible that there may be an analogous result if we consider a slightly larger ideal I, which we hope to investigate in a forthcoming paper.

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