DECORATED TANGLES AND CANONICAL BASES

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Abstract. We study the combinatorics of fully commutative elements in Coxeter groups of type H_n for any $n > 2$. Using the results, we construct certain canonical bases for non-simply-laced generalized Temperley–Lieb algebras and show how to relate them to morphisms in the category of decorated tangles.

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Introduction

The category of decorated tangles was introduced by the author [**4**] using ideas of Martin and Saleur [**12**]. This allows a generalization of the well-known diagram calculus [**13**] for the Temperley–Lieb algebra to Coxeter systems of other types. We showed in [**4**] and [**5**] how the endomorphisms in the category of decorated tangles may be used to construct faithful representations of generalized Temperley–Lieb algebras arising from Coxeter systems of types B, D or H. By realizing these algebras, which are quotients of Hecke algebras, in terms of the category, we can prove results about their representation theory and structure which may otherwise not be obvious, particularly in the case of type H.

In [**6**], it was shown that a generalized Temperley–Lieb algebra arising from a Coxeter system of arbitrary type admits a canonical basis (IC basis). Formally, this is similar to the basis $\{C'_w : w \in W\}$ for the Hecke algebra introduced by

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Kazhdan and Lusztig [**10**], although the precise relationship between the two is not completely obvious.

The goal of this paper is to tie these two theories together by showing how decorated tangles may be used to describe certain canonical bases for generalized Temperley–Lieb algebras. This is easily done in types D (and A) by using [**6**, Theorem 3.6] and [**4**, Theorem 4.2], so we concentrate here on the nontrivial cases of Coxeter systems of types B and H.

According to [**1**], interesting algebras and representations defined over $\mathbb{N}$ come from category theory, and are best understood when their categorical origin has been discovered. It is conjectured [**7**, Conjecture 1.2.4] that the canonical basis in the preceding paragraph should have structure constants in $\mathbb{N}[v, v^{-1}]$, and the results of this paper may be regarded as discovering the categorical origin of this phenomenon in certain special cases.

Further motivation for our results is as follows. In type B, it is known from [**7**, Theorem 2.2.1] that the canonical basis is the projection of a certain subset of the Kazhdan–Lusztig basis, but no purely combinatorial construction is given; we give such a construction here (Theorem 2.2.5). Conversely, in type H, we have a combinatorial construction of a basis with very convenient and useful properties [**5**, §4], but we have no simple abstract characterisation of the basis; we show here that the basis of decorated tangles from [**5**] is identical to the canonical basis (Theorem 2.1.3).

Our methods of proof are combinatorial, and a by-product of the proof is a construction of the canonical basis in types B and H which avoids diagrams and Kazhdan–Lusztig theory and uses only the combinatorics of fully commutative words (see Theorem 3.4.3 and Theorem 5.2.1). It is therefore necessary to develop an understanding of this combinatorics in type H, which we do in §3. As we see in §5, the combinatorics in type H behaves rather like a superset of the combinatorics of fully commutative expressions in type B; the latter has been studied extensively by Stembridge [**15**, §§2–6] from a very different perspective.

Apart from proving the claim in [**6**, Remark 2.4 (1)], the results of this paper are of interest because of their applications. For example, our results here are related in a very precise way to Jones' planar algebras [**9**]; this is the subject of a future paper. Furthermore, there are applications to Kazhdan–Lusztig theory:

the curious combinatorial rules in definitions 2.1.1 and 2.2.4 can be interpreted to give a combinatorially explicit description of those Kazhdan–Lusztig cells that contain fully commutative elements, certainly when the associated Coxeter groups are finite, and conjecturally in general.

1. Preliminaries

1.1 Decorated tangles.

We start by recalling the definition of the category of decorated tangles which was introduced in [**4**]. This is a mild generalization of Martin and Saleur's diagram calculus for the blob algebra in [**12**].

Definition 1.1.1. A tangle is a portion of a knot diagram contained in a rectangle. The tangle is incident with the boundary of the rectangle only on the north and south faces, where it intersects transversely. The intersections in the north (respectively, south) face are numbered consecutively starting with node number 1 at the western (i.e., the leftmost) end.

Two tangles are *equivalent* if there exists an isotopy of the plane carrying one to the other such that the corresponding faces of the rectangle are preserved setwise. Two tangles are *vertically equivalent* if they are equivalent in the above sense by an isotopy which preserves setwise each vertical cross-section of the rectangle.

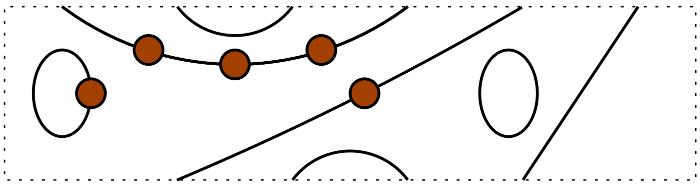
We call the edges of the rectangular frame "faces" to avoid confusion with the "edges" which are the arcs of the tangle.

Definition 1.1.2. A decorated tangle is a crossing-free tangle in which an arc exposed to the west face of the rectangular frame is allowed to carry decorations. (Any arc not exposed to the west face of the rectangular frame must be undecorated.)

By default, the decorations will be discs, although we will need two kinds of decorations for some of our later applications.

Any decorated tangle consists only of loops and edges, none of which intersect each other. A typical example is shown in Figure 1.

FIGURE 1. A decorated tangle



Definition 1.1.3. The category of decorated tangles, $\mathbb{DT}$, has as its objects the natural numbers. The morphisms from n to m are the equivalence classes of decorated tangles with n nodes in the north face and m in the south. The source of a morphism is the number of points in the north face of the bounding rectangle, and the target is the number of points in the south face. Composition of morphisms works by concatenation of the tangles, matching the relevant south and north faces together.

Note that for there to be any morphisms from n to m, it is necessary that $n+m$ be even.

We now define the algebra of decorated tangles.

Definition 1.1.4. Let R be a commutative ring with 1 and let n be a positive integer. Then the R-algebra $\mathbb{DT}_n$ has as a free R-basis the morphisms from n to n, where the multiplication is given by the composition in $\mathbb{DT}$.

Definition 1.1.5. The edges in a tangle T which connect nodes (i.e., not the loops) may be classified into two kinds: propagating edges, which link a node in the north face with a node in the south face, and non-propagating edges, which link two nodes in the north face or two nodes in the south face.

1.2 Canonical bases.

Let X be a Coxeter graph, of arbitrary type, and let $W(X)$ be the associated Coxeter group with distinguished set of generating involutions $S(X)$. Denote by $\mathcal{H}(X)$ the Hecke algebra associated to $W(X)$. (A good reference for Hecke algebras is [**8**, §7].) Let $\mathcal{A}$ be the ring of Laurent polynomials, $\mathbb{Z}[v, v^{-1}]$. The $\mathcal{A}$-algebra $\mathcal{H}(X)$ has a basis consisting of elements T_w, with w ranging over $W(X)$, that satisfy

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where ℓ is the length function on the Coxeter group $W(X)$, $w \in W(X)$, and $s \in S(X)$. The parameter q is equal to v^2.

Let $J(X)$ be the two-sided ideal of $\mathcal{H}(X)$ generated by the elements

$$\sum_{w \in \langle s, s' \rangle} T_w,$$

where (s, s') runs over all pairs of elements of $S(X)$ that correspond to adjacent nodes in the Coxeter graph. (If the nodes corresponding to (s, s') are connected by a bond of infinite strength, then we omit the corresponding relation.)

Definition 1.2.1. Following Graham [**3**, Definition 6.1], we define the generalized Temperley–Lieb algebra $TL(X)$ to be the quotient $\mathcal{A}$-algebra $\mathcal{H}(X)/J(X)$. We denote the corresponding epimorphism of algebras by $\theta : \mathcal{H}(X) \longrightarrow TL(X)$.

Definition 1.2.2. A product $w_1 w_2 \cdots w_n$ of elements $w_i \in W(X)$ is called *reduced* if $\ell(w_1 w_2 \cdots w_n) = \sum_i \ell(w_i)$. We reserve the terminology *reduced expression* for reduced products $w_1 w_2 \cdots w_n$ in which every $w_i \in S(X)$.

Call an element $w \in W(X)$ *complex* if it can be written as a reduced product $x_1 w_{ss'} x_2$, where $x_1, x_2 \in W(X)$ and $w_{ss'}$ is the longest element of some rank 2 parabolic subgroup $\langle s, s' \rangle$ such that s and s' correspond to adjacent nodes in the Coxeter graph.

Denote by $W_c(X)$ the set of all elements of $W(X)$ that are not complex. The *fully commutative* elements of $W(X)$ are defined in [**14**] to be those elements $w \in W$ for which any reduced expression can be transformed into any other by repeatedly applying short braid relations, i.e., by iterated commutation of pairs of consecutive generators in the expression. The elements $W_c(X)$ are precisely the fully commutative elements of $W(X)$ by [**14**, Proposition 1.1].

We define the *content* of $w \in W$ to be the set $c(w)$ of Coxeter generators $s \in S$ that appear in a reduced expression for w. This can be shown not to depend on the reduced expression chosen by using the theory of Coxeter groups, e.g. by applying Matsumoto's Theorem.

Let t_w denote the image of the basis element $T_w \in \mathcal{H}(X)$ in the quotient $TL(X)$.

Theorem 1.2.3 (Graham). *The set $\{t_w : w \in W_c\}$ is an $\mathcal{A}$-basis for the algebra $TL(X)$.*

Proof. See [**3**, Theorem 6.2]. $\square$

Definition 1.2.4. For each $s \in S(X)$, we define $b_s = v^{-1}t_s + v^{-1}t_e$, where e is the identity element of W. If $w \in W_c(X)$ and $s_1 s_2 \cdots s_n$ is a reduced expression for w, then we define $b_w = b_{s_1} b_{s_2} \cdots b_{s_n}$. We take the empty product b_e to be the identity element t_e of $TL(X)$.

Note that b_w does not depend on the choice of reduced expression for w.

It is known that $\{b_w : w \in W_c\}$ is a basis for the $\mathcal{A}$-module $TL(X)$; this can be deduced from Theorem 1.2.3 by using [**6**, Lemma 1.5]. We shall call it the *monomial basis.*

We now recall a principal result of [**6**], which establishes the canonical basis for $TL(X)$. This basis is a direct analogue of the important Kazhdan–Lusztig basis of the Hecke algebra $\mathcal{H}(X)$.

Fix a Coxeter graph, X. Let $I = W_c(X)$, let $\mathcal{A}^- = \mathbb{Z}[v^{-1}]$, and let $\bar{\ }$ be the involution on the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ which satisfies $\bar{v} = v^{-1}$.

By [**6**, Lemma 1.4], the algebra $TL(X)$ has a $\mathbb{Z}$-linear automorphism of order 2 that sends v to v^{-1} and t_w to $t_{w^{-1}}^{-1}$. We denote this map also by $\bar{\ }$.

Let $\mathcal{L}$ be the free $\mathcal{A}^-$-submodule of $TL(X)$ with basis $\{\widetilde{t}_w : w \in W_c\}$, where $\widetilde{t}_w := v^{-\ell(w)} t_w$, and let $\pi : \mathcal{L} \longrightarrow \mathcal{L}/v^{-1}\mathcal{L}$ be the canonical projection.

Theorem 1.2.5. *There exists a unique basis $\{c_w : w \in W_c\}$ for $\mathcal{L}$ such that $\overline{c_w} = c_w$ and $\pi(c_w) = \pi(\widetilde{t}_w)$ for all $w \in W_c$.*

Proof. This is [**6**, Theorem 2.3]. □

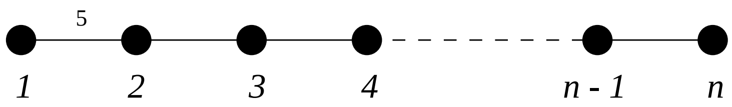
The basis $\{c_w : w \in W_c\}$ is called the *IC basis* (or the *canonical basis*) of $TL(X)$. It depends on the t-basis, the involution $\bar{\ }$, and the lattice $\mathcal{L}$.

2. Main results

2.1 Type H.

A Coxeter system of type H_n is given by the Coxeter graph in Figure 2.

FIGURE 2. Coxeter graph of type H_n



The algebras $TL(H_n)$ were first studied by Graham in [**3**, §7]. The basic properties of these algebras are also described in [**5**], to which the reader is referred for further explanatory comments and examples. We summarise some of the more important properties here.

The unital algebras $TL(H_n)$ over $\mathcal{A}$ can be defined by generators $b_1, b_2, \dots b_n$ (as in Definition 1.2.4) and relations

$$\begin{aligned} b_i^2 &= \delta b_i, \\ b_i b_j &= b_j b_i \quad \text{if} \quad |i-j| > 1, \\ b_i b_j b_i &= b_i \quad \text{if} \quad |i-j| = 1 \quad \text{and} \quad i, j > 1, \\ b_i b_j b_i b_j b_i &= 3 b_i b_j b_i - b_i \quad \text{if} \quad \{i, j\} = \{1, 2\}, \end{aligned}$$

where $\delta := [2] = v + v^{-1}$. All the algebras $TL(H_n)$ are finite dimensional, even though the Coxeter group $W(H_n)$ is infinite for $n > 4$.

The algebras $TL(H_n)$ can also be described in terms of decorated tangles; for this, we need the concept of an H-admissible diagram from [**5**, Definition 2.2.1].

Definition 2.1.1. An H-admissible diagram with n edges is an element of $\mathbb{DT}_n$ with no loops which satisfies the following conditions.

(i) No edge may be decorated if all the edges are propagating.

(ii) If there are non-propagating edges in the diagram, then either there is a decorated edge in the north face connecting nodes 1 and 2, or there is a non-decorated edge in the north face connecting nodes i and $i+1$ for some $i > 1$. A similar condition holds for the south face.

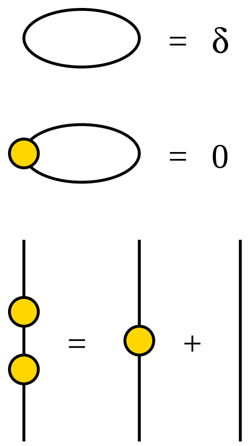
(iii) Each edge carries at most one decoration.

The H-admissible diagram U_i, where $1 \leq i \leq n$, is the diagram all of whose edges are propagating and undecorated, except for those attached to nodes i and $i+1$ in the north face, and nodes i and $i+1$ in the south face. These four nodes are connected in the pairs given, using decorated edges if $i = 1$, and using undecorated edges if $i > 1$.

The main result of [**5**] is the following.

Theorem 2.1.2. *Let Δ_n^H be the $\mathcal{A}$-algebra with basis given by the H-admissible diagrams with n edges and multiplication induced from the multiplication on $\mathbb{DT}_n$, subject to the reduction rules in Figure 3. Then the map sending b_i to U_i extends to an isomorphism of $\mathcal{A}$-algebras between $TL(H_{n-1})$ and Δ_n^H.*

FIGURE 3. Reduction rules for Δ_n



Proof. This is [**5**, Theorem 3.4.2]. □

Recall [**5**, Remark 2.2.3] that the first two relations in Figure 3 determine which scalar to multiply by when a loop is removed, and the third relation expresses a diagram as a sum of two other diagrams with fewer decorations.

One of the two main results of this paper is the next result.

Theorem 2.1.3. *The diagram basis described in Theorem 2.1.2 is the canonical basis of* $TL(H_{n-1})$ *in the sense of Theorem 1.2.5.*

2.2 Type B.

We have results analogous to those in §2.1 for Coxeter systems of type B. Here, nodes 1 and 2 in the Coxeter graph are connected by a bond of strength 4 rather than 5.

The finite dimensional, unital algebras $TL(B_n)$ over $\mathcal{A}$ can be defined by generators $b_1, b_2, \ldots, b_n$ and relations

$$\begin{aligned} b_i^2 &= \delta b_i, \\ b_i b_j &= b_j b_i \quad \text{if } |i-j| > 1, \\ b_i b_j b_i &= b_i \quad \text{if } |i-j| = 1 \text{ and } i, j > 1, \\ b_i b_j b_i b_j &= 2 b_i b_j \quad \text{if } \{i, j\} = \{1, 2\}, \end{aligned}$$

where $\delta := [2] = v + v^{-1}$.

We now describe $TL(B_n)$ in terms of decorated tangles. We first recall the description from [**4**], and then show how it can be adapted to give a result analogous to Theorem 2.1.3. It is convenient for this purpose to introduce some further terminology to classify tangles.

Definition 2.2.1. We say an edge in a tangle is of type p_1 if it connects node 1 in the north face to node 1 in the south face. We say the edge is of type p_3 if it does not involve either node 1 in the north face or node 1 in the south face. Otherwise, we say the edge is of type p_2.

The following definition is compatible with that in [**4**, Theorem 4.1].

Definition 2.2.2. A B-admissible diagram with n edges is an element of $\mathbb{DT}_n$ with no loops which satisfies one of the following three mutually exclusive conditions:

(B1) There is an undecorated edge of type p_1 (and no other decorations).

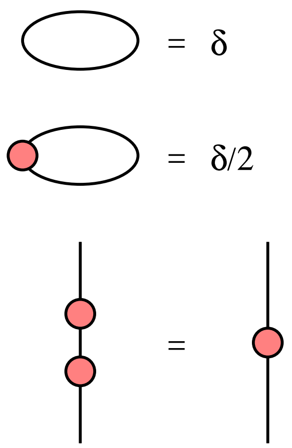
(B1′) There is a decorated edge of type p_1 and at least one non-propagating edge.

(B2) There are two decorated edges of type p_2.

We recall one of the main results of [**4**], noting that the diagrams U_i are B-admissible.

Theorem 2.2.3. *Let Δ_n^B be the $\mathbb{Q}[v, v^{-1}]$-algebra with basis given by the B-admissible diagrams with n edges and multiplication induced from the multiplication on $\mathbb{DT}_n$, subject to the reduction rules in Figure 4. Then the map sending b_1 to $2U_1$ and b_i to U_i, for $i > 1$, extends to an isomorphism of $\mathbb{Q}[v, v^{-1}]$-algebras between $TL(B_{n-1})$ and Δ_n^B.*

FIGURE 4. Reduction rules for type B

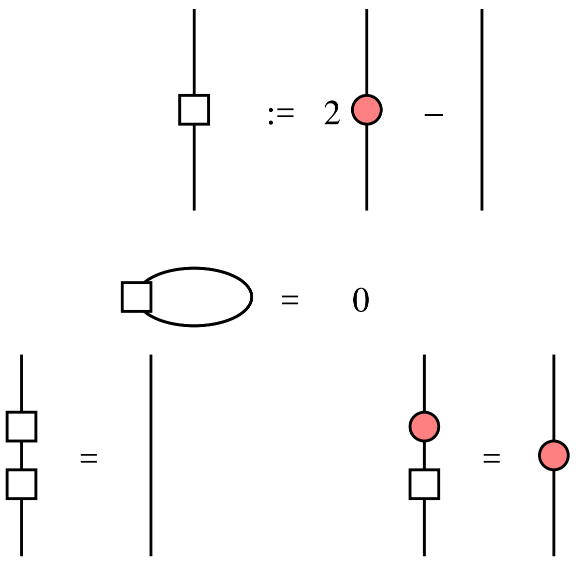


Proof. See [**4**, Theorem 4.1]. □

The reason for the presence of $\mathbb{Q}$ rather than $\mathbb{Z}$ in Theorem 2.2.3 is because our integral form of the algebra $TL(B_{n-1})$ is not the same as the integral form used in [**4**]. We will return to this issue in §5.1.

The basis in Theorem 2.2.3 is not the canonical basis, although it is closely related to it. In order to describe the canonical basis combinatorially, it is convenient to introduce a second type of decoration, denoted by a square. This satisfies the relations in Figure 5. The first rule given is the definition of the square decoration, and the others are reduction rules which are immediate consequences of the definition and the relations in Figure 4; we shall use these freely without further comment.

FIGURE 5. The square decoration



Definition 2.2.4. A B-canonical diagram with n edges is an element of $\mathbb{DT}_n$ with no loops that satisfies one of the following three mutually exclusive conditions C1, C1′, C2 below. All the decorations are square, except for those on the two edges involved in C2.

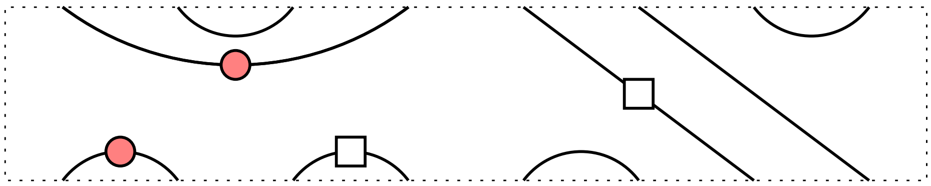
(C1) There is an undecorated edge of type p_1 (and no other decorations).

(C1′) There is an edge of type p_1 with a square decoration, and at least one non-propagating edge.

(C2) There are two edges of type p_2 with circular decorations.

We denote by C_n^1 the set of all elements of $\mathbb{DT}_n$ that satisfy C1 or C1′, and we denote by C_n^2 the set of elements of $\mathbb{DT}_n$ that satisfy C2. The example in Figure 6 is an element of C_8^2.

FIGURE 6. A B-canonical diagram



Our second main result is the following.

Theorem 2.2.5. *The set $C_n := C_n^1 \cup \{2D : D \in C_n^2\}$ is the canonical basis of $TL(B_{n-1})$ in the sense of Theorem 1.2.5.*

3. Combinatorial construction of the basis in type H.

The aim of §3 is to obtain an explicit, elementary description of the canonical basis in type H in terms of the monomial basis $\{b_w : w \in W_c\}$ (see Definition 1.2.4). The answer turns out to be more complicated than in the finite dimensional simply-laced case, where the basis may be described in terms of certain monomials [**6**, Theorem 3.6].

In §3, all computations take place in $TL(H_{n-1})$ over the ring $\mathcal{A}$ unless otherwise stated.

3.1 Deletion properties for monomials in type H.

Lemma 3.1.1. *The $\mathbb{Z}$-span of the monomial basis of $TL(H_{n-1})$ is equal to the $\mathbb{Z}$-span of the diagram basis of $TL(H_{n-1})$ described in Theorem 2.1.2.*

Proof. For the purposes of the proof, we shift base rings and regard $TL(H_{n-1})$ as the $\mathbb{Z}[\delta]$-algebra generated by the monomial basis elements b_i. (The generators and relations in §2.1 and the relations in the diagram algebra show that this makes sense.) Denote by $TL_\delta = TL_\delta(H_{n-1})$ the $\mathbb{Z}[\delta]$-version of the algebra.

From Definition 1.2.4, we know that $\{b_w : w \in W_c\}$ is an $\mathcal{A}$-basis for $TL(H_{n-1})$. We claim that it is a $\mathbb{Z}[\delta]$-basis for the version of the algebra we consider here. To prove this, it suffices to show that the set given is a spanning set. Let $a \in TL_\delta$. Then, since a is fixed by $^-$ (since δ and b_i are), we know that when a is expressed as a linear combination of b_w (with coefficients in $\mathcal{A}$), the coefficients are fixed by $^-$. An easy induction on the degree of Laurent polynomials shows that the elements of $\mathcal{A}$ fixed by $^-$ are precisely the elements of $\mathbb{Z}[\delta]$, which completes the claim.

A similar argument shows that the diagram $\mathcal{A}$-basis elements lie in TL_δ, because they are $\mathbb{Z}$-linear combinations of monomials in the b_i by [**5**, Proposition 3.2.8] and therefore fixed by $\bar{\ }$.

Let $b_w = \sum_x \lambda_x a_x$, where the a_x are diagram basis elements. To complete the proof, it suffices to show that all the λ_x are integers. The reduction rules in Figure 3 show that this will happen if no loops appear in the diagram corresponding to the monomial b_w. If k loops appear, the rules show that λ_x will equal $\delta^k z$ for some (possibly zero) integer z. Since the b_w form a $\mathbb{Z}[\delta]$-basis for TL_δ and δ is not invertible in $\mathbb{Z}[\delta]$, we must have $k = 0$ in each case. This completes the claim. $\square$

The next definition will turn out (once we have proved Theorem 3.4.3) to agree with the lattice $\mathcal{L}$ of §1.2.

Definition 3.1.2. The lattice $\mathcal{L}_H$ is the free $\mathcal{A}^-$-module on the monomial basis (or diagram basis) of $TL(H_{n-1})$. Let $\pi_H : \mathcal{L}_H \longrightarrow \mathcal{L}_H / v^{-1}\mathcal{L}_H$ be the canonical projection.

The equivalence of the two versions of the definition is immediate from Lemma 3.1.1.

Recall from Definition 1.2.4 that $b_e = \widetilde{t}_e$ and $b_i = \widetilde{t}_i + v^{-1}\widetilde{t}_e$. The following result, which is analogous to [**6**, Lemma 3.3], is of key importance in the sequel.

Lemma 3.1.3. *Let $b = b_{i_1} b_{i_2} \cdots b_{i_k}$ be an arbitrary monomial in the generators $b_i (1 \leq i < n)$. Let $\widehat{b}(l)$ denote the monomial obtained from b by omission of the l-th term, b_{i_l}. Then if $b \in v^m \mathcal{L}_H$ for some integer m, we have $\widehat{b}(l) \in v^{m+1}\mathcal{L}_H$.*

Proof. Consider the (not necessarily H-admissible) diagram corresponding to the monomial b. This has a certain number, c, of loops. It is clear from the nature of the diagrams that removing one generator b_{i_l} will change the number of loops by at most 1, so the number of loops, $\widehat{c}$, in $\widehat{b}(l)$ satisfies $\widehat{c} \leq c + 1$. Expanding $\widehat{b}(l)$ in terms of the diagram basis and applying Lemma 3.1.1 shows that $\widehat{b}(l) \in v^{m+1}\mathcal{L}_H$, as required. $\square$

It is convenient to introduce some terminology to describe the individual entries in a monomial in the generators b_i. We are particularly interested in the case of monomials which correspond to fully commutative reduced words (as in Definition 1.2.4), but we will also need the terminology in more general situations. We refer below to the instances of the individual generators b_i in a monomial as "letters", for the sake of clarity.

Definition 3.1.4. Let $b = b_{i_1} b_{i_2} \cdots b_{i_k}$ be a monomial in the generators b_i. Let $p = i_l$ for some l.

We say the letter b_p is *internal* if, after applying commutations, the monomial b may be transformed to a monomial in which the (same occurrence of the) letter b_p occurs as the middle term of a sequence $b_q b_p b_q$ for some letter b_q that does not commute with b_p. Otherwise, we say the letter b_p is *external*.

We say the letter b_p is *lateral* (with respect to an internal letter b_q) if it is external and, after commutations, it can be moved adjacent to the internal letter b_q to form a subsequence $b_p b_q b_p$, where b_p and b_q do not commute. We say the letter b_p is *bilateral* if it is lateral with respect to two different internal letters.

We will also apply these definitions in the obvious way to reduced expressions for fully commutative elements.

Remark 3.1.5. An important fact is that if $b = b_w$ is an element of the monomial basis, then the only possible internal letters are occurrences of b_1 and b_2.

To illustrate these concepts, here are some examples.

Example 3.1.6. Consider the fully commutative reduced expression

$$b = b_{i_1} \cdots b_{i_7} := b_1 b_2 b_3 b_1 b_2 b_1 b_2$$

in $TL(H_3)$. The letters b_{i_2}, b_{i_5} and b_{i_6} are internal, and the others are external. Of the external letters, b_{i_1}, b_{i_4} and b_{i_7} are lateral, and of these, b_{i_4} is the only bilateral one.

It is important to identify certain configurations that are not allowed to occur in the diagrammatic representation of a monomial basis element. The short dashed lines in the figures indicate that the two curves are part of the same generator b_i in a tangle for the monomial.

Figure 7. Some impermissible configurations

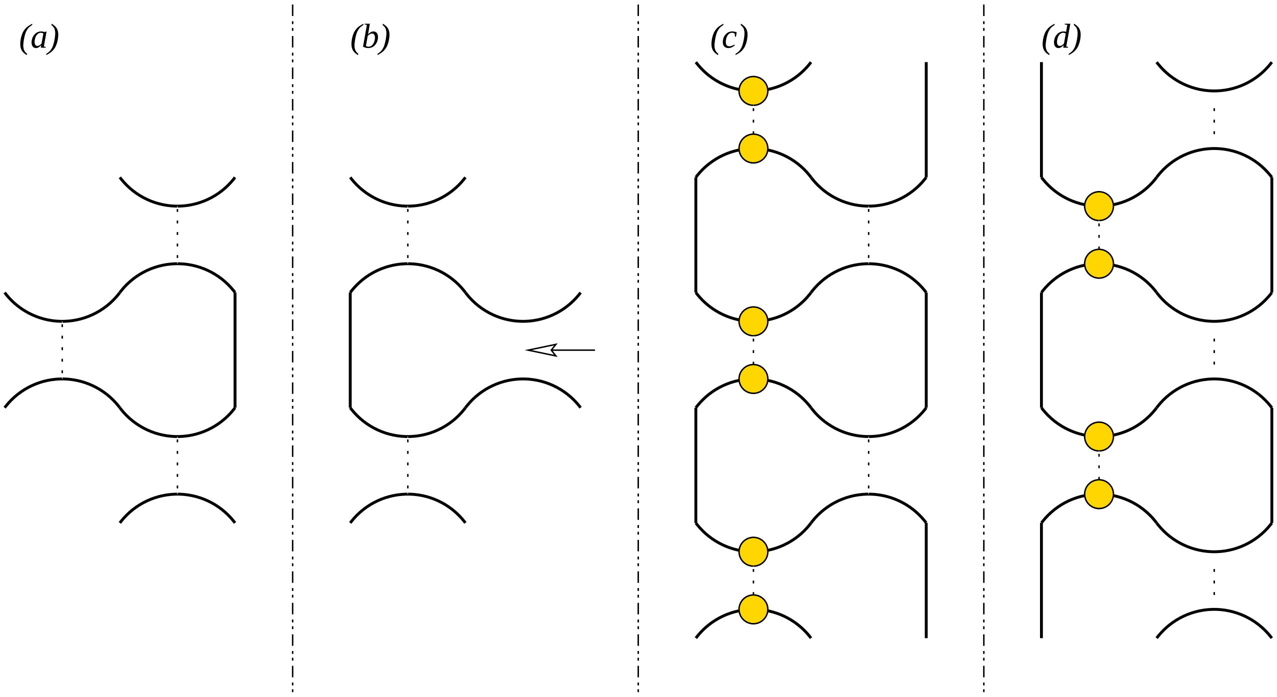


Figure 8. Impermissible configuration (e)

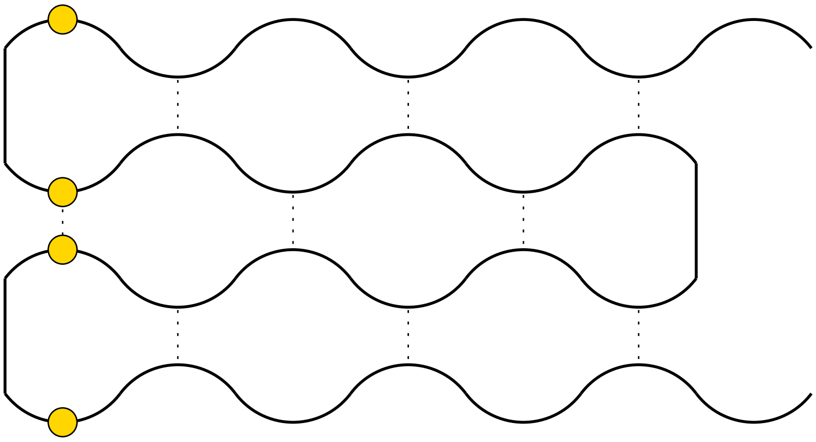
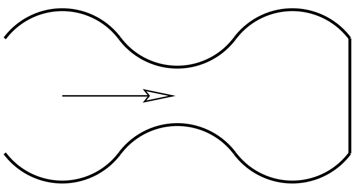


Figure 9. Impermissible configuration (f)



Lemma 3.1.7. *Consider a tangle T_V which is vertically equivalent to a monomial b_w for $w \in W_c$. Then T_V does not contain any segments vertically equivalent to those in parts (a) to (d) of Figure 7.*

Proof. Note first that commutation of generators $b_i b_j = b_j b_i$ preserves vertical equivalence. Parts (a), (c) and (d) of Figure 7 correspond, after applying commutations if necessary, to sequences $b_{i+1}b_ib_{i+1}$ (for some $i > 1$), $b_1b_2b_1b_2b_1$ and $b_2b_1b_2b_1b_2$, none of which can occur in a fully commutative monomial by definition. (Compare with "Property R" depicted in Figure 2 of [**2**].)

For (b), either the configuration corresponds to a sequence $b_ib_{i+1}b_i$ (again for $i > 1$) or there is a smaller configuration of the same shape as (b) inserted in the position shown by the arrow. Iterating this argument rightwards shows that there is an occurrence of $b_ib_{i+1}b_i$ (for some $i > 1$) somewhere, which is not allowed. □

Remark 3.1.8. Figure 8 shows an impossible situation which violates condition (a) of Figure 7. Similar counterexamples exist where the lobes of the arc are either longer or shorter than the one shown in the diagram.

The situation in Figure 9 is also impermissible (compare condition (b) of Figure 7). In this situation, there is either a smaller configuration of the same shape as (f) inserted in the position shown by the arrow, or we have a violation of condition (a), (b) or (c) of Figure 7.

These facts are very important in the proof of the next result, which is a more specialized version of Lemma 3.1.3.

Proposition 3.1.9. *Let $b = b_w \in \mathcal{L}_H$ be an element of the monomial basis. Let b_p be a letter in the monomial b, and let $\widehat{b}$ be the monomial obtained from b by deleting the letter b_p. Suppose $\widehat{b} \notin \mathcal{L}_H$. Then w can be parsed in one of the following ways, where the deleted letter is the barred one:*

(i) $w = w_1s_1s_3\cdots s_{2k-1}s_2s_4\cdots\overline{s_{2k}}s_1s_3\cdots s_{2k-1}w_2$, *where* $p = 2k > 2$,

(ii) $w = w_1s_1s_3\cdots s_{2k+1}s_2s_4\cdots s_{2k}s_1s_3\cdots s_{2k-1}w_2\overline{s_{2k}}s_{2k+1}w_3$, *where* $p = 2k$ *and* s_{2k+1} *commutes with every member of* $c(w_2)$,

(iii) $w = w_1s_{2k+1}\overline{s_{2k}}w_2s_1s_3\cdots s_{2k-1}s_2s_4\cdots s_{2k}s_1s_3\cdots s_{2k+1}w_3$, *where* $p = 2k$ *and* s_{2k+1} *commutes with every member of* $c(w_2)$,

(iv) $w = w_1s_q\overline{s_p}s_qw_2$, *where* $\{p, q\} = \{1, 2\}$.

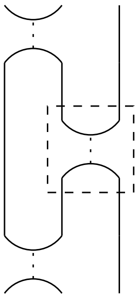
Conversely, if w can be parsed in one of the ways above, removal of the barred letter results in a monomial $\widehat{b} \notin \mathcal{L}_H$.

Definition 3.1.10. We say the letter $\overline{s_p}$ in w is critical of type (i), type (ii) or type (iii) respectively if, after commutations, w can be parsed as in parts (i), (ii) or (iii) respectively of Proposition 3.1.9. (If $\overline{s_p}$ satisfies condition (iv), this is the same as saying $\overline{s_p}$ is internal.)

Proof of Proposition 3.1.9. We proceed using the diagram calculus, by considering an element T_V vertically equivalent to the tangle corresponding to the monomial b_w.

Recall that none of the configurations in figures 7 and 8 is allowed to occur. If the letter to be deleted is internal as in case (iv), the situation is as shown in Figure 10.

FIGURE 10. Deleting a letter



Suppose we are not in case (iv). A routine, but nontrivial, case by case check using Lemma 3.1.7 and Remark 3.1.8 shows that if the letter being deleted forms a loop to the left, we must be in the situation shown in Figure 11. If, on the other hand, the letter being deleted forms a loop to the right, we must be in the situation shown in Figure 12 or its top-bottom mirror image. (As in Figure 8, the lobes of the arcs occurring may be longer or shorter than those shown.)

FIGURE 11. Deleting a critical letter of type (i)

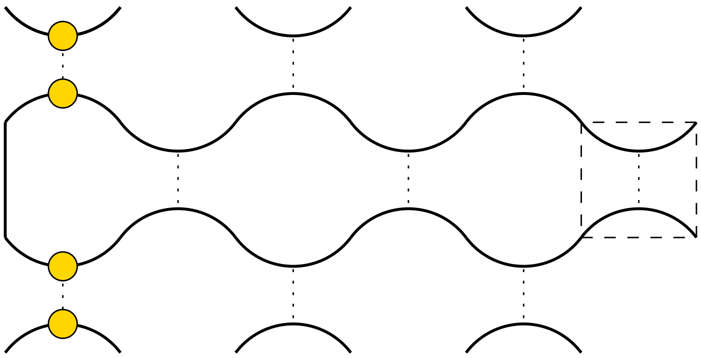


FIGURE 12. Deleting a critical letter of type (ii)

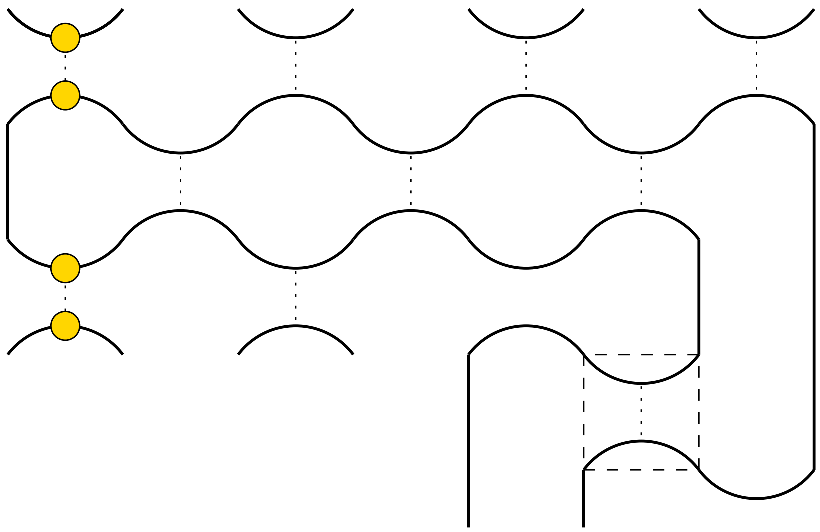


Figure 11 corresponds to case (i), Figure 12 corresponds to case (ii) and the top-bottom mirror image of Figure 12 corresponds to case (iii). (Note that in Figure 12, the two loose ends just to the left of the generator to be removed are necessarily connected to the south face, by straight lines.)

The converse statement, that removal of the barred letter in each case produces an extra loop, is immediate from the diagrams shown. □

Remark 3.1.11. Consider the special case of Proposition 3.1.9 where the letter b_p to be deleted satisfies $p \in \{1, 2\}$. Here, there are two possible cases: (a) that the letter b_p is internal or (b) that we have deleted an occurrence of b_2 which, after commutations, can be made to appear as the overscored letter in a subsequence of the form $b_3 b_1 b_2 b_1 \overline{b_2} b_3$ or $b_3 \overline{b_2} b_1 b_2 b_1 b_3$. In case (b), we say the corresponding occurrence of s_2 is *bad*. Note that all bad occurrences of s_2 are also lateral.

3.2 The f-basis.

The aim of §3.2 is to define a basis of polynomials in the generators b_i which will eventually be seen to equal the canonical basis in type H.

Lemma 3.2.1. *In a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ for $w \in W_c$, a bilateral letter in the expression can only be an occurrence of s_1.*

Proof. Let $s = s_{i_k}$ be a bilateral letter in the reduced expression. Since s is lateral, we can apply commutations so that it lies in a subsequence sts where t is internal. Since $w \in W_c$, we must have $\{s, t\} = \{s_1, s_2\}$, so it remains to show that $s \neq s_2$.

A bilateral occurrence of s_2 would have to occur in a subsequence $s_1 w' s_2 w'' s_1$, where there are exactly two occurrences of s_1, both internal, and w' and w'' are (reduced) words in the generators that commute with s_2. Since the expression is reduced, w' and w'' cannot contain occurrences of s_2. It follows that w' and w'' commute with s_1, which implies that the occurrence of s_2 is internal, contrary to the hypothesis that it is bilateral. $\square$

We now state some results on the parsing of reduced expressions in type H for later usage.

Lemma 3.2.2. *Let $w \in W_c$, and let $s_{i_1} s_{i_2} \cdots s_{i_l}$ be an arbitrary (but fixed) reduced expression for w.*

(i) *Suppose the expression can be parsed as $w = w_1 s w_2 t w_3 s w_4 t w_5$, where $\{s, t\} = \{s_1, s_2\}$. Suppose further that w also has reduced expressions of the forms*

$$w = w_1 w_2 sts w_3 w_4 t w_5$$

and

$$w = w_1 s w_2 w_3 tst w_4 w_5.$$

Then there is a reduced expression for w for which all four occurrences of s and t occur consecutively.

(ii) *It is impossible to have three consecutive internal letters in the reduced expression.*

Proof. We first prove (i). The second and third hypotheses respectively guarantee that s commutes with every member of $c(w_2) \cup c(w_3)$ and that t commutes with every member of $c(w_3) \cup c(w_4)$. We can therefore transform the original reduced expression as follows:

$$\begin{aligned} w &= w_1 s w_2 t w_3 s w_4 t w_5 \\ &= w_1 w_2 s t w_3 s t w_4 w_5 \\ &= w_1 w_2 w_3 stst w_4 w_5. \end{aligned}$$

This completes the proof of (i).

To prove (ii), note first that three consecutive internal letters would have to be of the form sts, where $\{s, t\} = \{s_1, s_2\}$. Since both occurrences of s are internal, the sts subsequence must occur in a longer subsequence of the form $tw' sts w'' t$, where w' and w'' are words in the generators that commute with t. By applying commutations, we can form a subsequence of the form $tstst$, contrary to $w \in W_c$. □

Definition 3.2.3. Let $w \in W_c$. Let R be the set of internal letters, s, of w that have the property that, after suitable commutations, they occur in a subsequence tst where the rightmost occurrence of t is bilateral.

Fix a reduced expression for $w \in W_c$. We say this expression is *right justified* if (without applying commutations):

(i) for all letters s in the set R, the letter immediately to the left is either lateral or internal and

(ii) for all other internal letters s, both the letter immediately to the left and the letter immediately to the right are either lateral or internal.

Example 3.2.4. Let $w = s_1s_2s_3s_1s_2s_1s_2 \in W(H_3)$ (compare with Example 3.1.6). The set R has one element, namely the leftmost occurrence of s_2.

The reduced expression $s_1s_2s_3s_1s_2s_1s_2$ is right justified, but the reduced expression $w = s_1s_2s_1s_3s_2s_1s_2$ is not: the occurrence of s_2 in third from right place fails condition (ii).

Right justified reduced expressions for a given $w \in W_c$ need not be unique, but they always exist, as we see below.

Lemma 3.2.5. *If $w \in W_c$, there exists a right justified reduced expression for w in which all internal and lateral letters occur in (maximal) contiguous subsequences of the form (1) $s_1\overline{s_2}$, (2) $s_1\overline{s_2}s_1$, (3) $s_2\overline{s_1}s_2$, (4) $s_1\overline{s_2s_1}s_2$, (5) $s_2\overline{s_1s_2}$, or (6) $s_2\overline{s_1s_2}s_1$. (The internal letters are the barred ones; all other letters are lateral.)*

Proof. We describe a simple algorithm to find such a reduced expression. Start with any reduced expression and work out which letters are in the set R. Apply commutations until each internal letter not in the set R occurs as the middle term in a subsequence sts. This produces an expression of the desired form: note that all the bilateral letters (which are occurrences of s_1 by Lemma 3.2.1) have been commuted to the right until they are adjacent to internal occurrences of s_2.

The assertion about the subsequences comes from the construction and from Lemma 3.2.2. Part (i) of that lemma explains why it is possible to form subsequences of types (4), (5) and (6) in the statement, and part (ii) explains why no longer kinds of subsequence can occur. $\square$

We can now define a polynomial over $\mathbb{Z}$ in the generators b_i using the preceding results.

Definition 3.2.6. Let $w \in W_c$, and let $s_{i_1} s_{i_2} \cdots s_{i_l}$ be a right justified reduced expression for w.

We obtain the polynomial $f_w \in TL(H_{n-1})$ by substituting the letters in w as follows. Each of the six subsequences described in Lemma 3.2.5 is replaced as follows:

$$\begin{aligned} s_1\overline{s_2} &\longrightarrow b_1b_2 - 1, \\ s_1\overline{s_2}s_1 &\longrightarrow b_1b_2b_1 - b_1, \\ s_2\overline{s_1}s_2 &\longrightarrow b_2b_1b_2 - b_2, \\ s_1\overline{s_2s_1}s_2 &\longrightarrow b_1b_2b_1b_2 - 2b_1b_2, \\ s_2\overline{s_1s_2} &\longrightarrow b_2b_1b_2 - 2b_2, \\ s_2\overline{s_1s_2}s_1 &\longrightarrow b_2b_1b_2b_1 - 2b_2b_1. \end{aligned}$$

All other s_i occurring are replaced by the corresponding generator b_i.

Note that any bilateral occurrence of s_1 must occur at the beginning of one of the contiguous subsequences listed in Lemma 3.2.5. We call such a subsequence *distinguished* if it begins with a bilateral occurrence of s_1. We call a factor of f_w obtained from a distinguished subsequence by the above replacement procedure a *distinguished factor*.

Example 3.2.7. Take the right justified reduced expression in Example 3.2.4. The corresponding polynomial is

$$f_w = (b_1b_2 - 1)b_3(b_1b_2b_1b_2 - 2b_1b_2).$$

The factor $(b_1b_2b_1b_2 - 2b_1b_2)$ is distinguished because it arises from a subsequence $s_1\overline{s_2s_1}s_2$ in which the first occurrence of s_1 is bilateral (see Example 3.1.6). The factors $(b_1b_2 - 1)$ and (b_3) are not distinguished.

Lemma 3.2.8.

(i) *The elements $\{f_w : w \in W_c\}$ are well-defined and form an $\mathcal{A}$-basis of $TL(H_{n-1})$.*

(ii) *For each $w \in W_c$ we have $f_w \in \mathcal{L}_H$.*

Proof. We tackle (i) first. The fact that the f_w are well-defined follows from Lemma 3.2.5: all internal and lateral elements end up in contiguous subsequences independent of the original reduced expression, and these are dealt with by Definition 3.2.6. All other letters go through a simple replacement procedure.

To see that the set is a basis, expand each f_w in terms of the monomial basis. The term b_w occurs with coefficient 1, and all other terms are sums of shorter monomials. Since an arbitrary monomial is (by using the relations in §2.1) a linear combination of basis monomials of equal or shorter length, we see that the set is a basis as claimed.

We now turn to (ii). Let $w \in W_c$. It is clear from the definition of f_w that it is a $\mathbb{Z}$-linear combination of monomials in the b_i, the longest of which is b_w and the others of which are obtained from b_w by deleting certain letters. From the diagram calculus, we see that the operations of (a) replacing a sequence $b_s b_t b_s$ by b_s and (b) replacing a sequence $b_s b_t b_s b_t$ by $b_s b_t$ (where $\{s, t\} = \{s_1, s_2\}$ in both cases) do not alter the number of loops. Since $b_w \in \mathcal{L}_H$ by Lemma 3.1.1, it follows that all the shorter monomials also lie in $\mathcal{L}_H$, and thus that $f_w \in \mathcal{L}_H$, as required. □

3.3 Treatment of internal letters.

In §3.3 we show that for each element f_w ($w \in W_c$), there is a certain monomial $\widehat{f_w}$ in the generators b_i and $\widetilde{t_i}$ which projects via π_H to the same element of the lattice $\mathcal{L}_H$. This is one of three main steps to prove that $\pi_H(f_w) = \pi_H(\widetilde{t_w})$.

Definition 3.3.1. Let b_w ($w \in W_c$) be an element in the monomial basis. The *expanded form* b'_w of b_w is the monomial obtained by replacing each bilateral occurrence of b_1 (see Lemma 3.2.1) by $b_1 b_1$.

We say a monomial b in the generators b_i is 1-commutative if it is equal as a monomial, after applying commutations, to b'_w for some $w \in W_c$.

Example 3.3.2. Let w be as in Example 3.2.4. Then $b = b_1 b_2 b_1 b_3 b_1 b_2 b_1 b_2$ is 1-commutative, since it is equal to $b_1 b_2 b_3 b_1^2 b_2 b_1 b_2$.

The next result is a version of Proposition 3.1.9 for the expanded forms of monomials.

Lemma 3.3.3. *Let $b = b'_w \in v^k \mathcal{L}_H$ be an expanded monomial. Let b_p be a letter in the monomial b, and let $\widehat{b}$ be the monomial obtained from b by deleting the letter b_p.*

Then $\widehat{b} \in v^k \mathcal{L}_H$ *unless* b_p *corresponds to an internal or critical letter of the basis monomial,* b_w*.*

Proof. There are two cases to consider. In the first case, the deleted letter is one of the doubled occurrences of b_1 arising from a bilateral occurrence of b_1 in b_w. In this case, removing the letter divides the monomial by δ. Since multiplication by δ increases the degree of a polynomial in v by 1, we see that deleting this occurrence of b_1 produces a monomial in $v^{k-1}\mathcal{L}_H$, from which the claim follows.

In the second case, the letter to be deleted is not bilateral. We proceed by first applying commutations and the relation $b_1^2 = \delta b_1$, obtaining $b'_w = \delta^l b_w$ for some l. The result then follows from Proposition 3.1.9. □

We can modify the f-basis in a similar way.

Definition 3.3.4. Let $f'_w \in TL(H_{n-1})$ be the element obtained by inserting an extra occurrence of b_1 immediately to the left of each distinguished factor (see Definition 3.2.6).

Example 3.3.5. Let f_w be as in Example 3.2.7. The underlying word w is given by $s_1 s_2 s_3 s_1 s_2 s_1 s_2$, and as we can see from Example 3.1.6, the fourth letter of this word is the only bilateral letter. Recall from Example 3.2.7 that the only distinguished factor in f_w is the rightmost factor in the expression

$$f_w = (b_1 b_2 - 1)(b_3)(b_1 b_2 b_1 b_2 - 2 b_1 b_2).$$

Inserting an occurrence of b_1 to the left of this factor now gives

$$f'_w = (b_1 b_2 - 1) b_3 b_1 (b_1 b_2 b_1 b_2 - 2 b_1 b_2)$$

or

$$f'_w = (b_1 b_2 - 1) b_1 b_3 (b_1 b_2 b_1 - 2 b_1) b_2.$$

Lemma 3.3.6. *We have the following identities:*

(i) $b_1 b_2 b_1 - b_1 = \widetilde{t}_1 \widetilde{t}_2 \widetilde{t}_1 + v^{-1} \widetilde{t}_2 \widetilde{t}_1 + v^{-1} v^{-1} \widetilde{t}_1 + \widetilde{t}_1 \widetilde{t}_2 v^{-1} + v^{-1} \widetilde{t}_2 v^{-1} + v^{-1} v^{-1} v^{-1}.$

(ii) $b_1 b_2 b_1 b_2 - 2 b_1 b_2 = (\widetilde{t}_1 \widetilde{t}_2 \widetilde{t}_1 \widetilde{t}_2 + \widetilde{t}_1 \widetilde{t}_2 \widetilde{t}_1 v^{-1}) + v^{-1} \widetilde{t}_2 \widetilde{t}_1 \widetilde{t}_2 + v^{-1} v^{-1} \widetilde{t}_1 \widetilde{t}_2 + v^{-1} v^{-1} v^{-1} \widetilde{t}_2 + v^{-1} \widetilde{t}_2 \widetilde{t}_1 v^{-1} + v^{-1} v^{-1} \widetilde{t}_1 v^{-1} + v^{-1} v^{-1} v^{-1} v^{-1}.$

There are two other similar identities obtained by exchanging the roles of 1 *and* 2 *above.*

Note. The reason the result has been expressed in such an inconcise way will become clear in the proof of Lemma 3.3.9.

Proof. This is a routine calculation. □

Definition 3.3.7. Let $w \in W_c$. The element $\widehat{f_w} \in TL(H_{n-1})$ is that obtained by taking the corresponding monomial basis element

$$b_w = b_{i_1} b_{i_2} \cdots b_{i_r}$$

and substituting $\widetilde{t_i}$ for b_i whenever b_i is an internal or lateral letter that is not a bad occurrence of b_2 (see Remark 3.1.11).

We define an expanded form $\widehat{f'_w}$ by replacing each occurrence of $\widetilde{t_1}$ in $\widehat{f_w}$ that corresponds to a bilateral occurrence of s_1 in w by $\widetilde{t_1}\widetilde{t_1}$.

Example 3.3.8. Let $w = s_1 s_2 s_3 s_1 s_2 s_1 s_2 s_3 \in W(H_3)$. In this case, the rightmost occurrence of s_2 is bad, and we have

$$\widehat{f_w} = \widetilde{t_1}\widetilde{t_2} b_3 \widetilde{t_1}\widetilde{t_2}\widetilde{t_1} b_2 b_3$$

and

$$\widehat{f'_w} = \widetilde{t_1}\widetilde{t_2} b_3 \widetilde{t_1}\widetilde{t_1}\widetilde{t_2}\widetilde{t_1} b_2 b_3.$$

It is more convenient for later purposes to move the new occurrence of $\widetilde{t_1}$ as far to the left as possible; this gives

$$\widehat{f'_w} = \widetilde{t_1}\widetilde{t_2}\widetilde{t_1} b_3 \widetilde{t_1}\widetilde{t_2}\widetilde{t_1} b_2 b_3.$$

Lemma 3.3.9. *For each $w \in W_c$, let $\kappa = \kappa(w)$ be the number of bilateral occurrences of s_1 in w. Then $\pi_H(v^{-\kappa} f'_w) = \pi_H(v^{-\kappa}\widehat{f'_w})$, where π_H is as in Definition 3.1.2.*

Proof. Consider f'_w, and replace all the sections corresponding (via Definition 3.2.6) to internal and lateral letters using the identities in Lemma 3.3.6. This shows that f'_w is equal to $\widehat{f'_w}$ plus lower monomial terms (in $\widetilde{t_i}$, b_i and v^{-1}) which differ from f'_w

in that certain of the $\widetilde{t}_i$ have been replaced by v^{-1}. Note that each term containing a v^{-1} is missing at least one lateral $\widetilde{t}_i$.

We now expand each of the lower monomial terms above to a polynomial in the b_i by using the identity $\widetilde{t}_i = b_i - v^{-1}$. This expression expands to a sum of monomials in the b_i, each of which can be obtained from the expanded form b'_w by (possibly) applying a sign change and then deleting some of the letters and replacing them with instances of v^{-1}. As in the previous paragraph, one of the missing letters is guaranteed to be lateral. Note that $b'_w = \delta^\kappa b_w \in v^\kappa \mathcal{L}_H$.

If the missing lateral letter is not a bad occurrence of b_2 then it is not internal or critical. In this case, we use Lemma 3.3.3 (applied to the missing lateral letter) and apply Lemma 3.1.3 repeatedly (if necessary) to show that all the lower monomial terms lie in $v^{\kappa-1}\mathcal{L}_H$. It follows that after multiplication by $v^{-\kappa}$ and applying π_H, the lower terms go to zero, which completes the argument.

If the missing lateral letter is a bad occurrence of b_2 then we proceed in the same way, but we are left with an extra term which will be $\widetilde{t}_1\widetilde{t}_2\widetilde{t}_1 v^{-1}$ or its left-right mirror image. The terms in Lemma 3.3.6 (ii) will all go to zero as before, except the two in parentheses; these can be combined to form $\widetilde{t}_1\widetilde{t}_2\widetilde{t}_1 b_2$ or its left-right mirror image; this is the term we require. □

Lemma 3.3.10. *For each $w \in W_c$, $\pi_H(f_w) = \pi_H(\widehat{f}_w)$.*

Proof. Let $\kappa = \kappa(w)$ as in Lemma 3.3.9. It is clear that $\pi_H(v^{-\kappa} f'_w) = \pi_H(f_w)$, so by Lemma 3.3.9, it is enough to prove that $\pi_H(v^{-\kappa}\widehat{f}'_w) = \pi_H(\widehat{f}_w)$.

Observe that $v^{-1}(\widetilde{t}_1)^2 = v^{-1} + (1 - v^{-2})\widetilde{t}_1$. We need to apply this relation κ times to $\widehat{f}'_w$. Arguing as in the proof of Lemma 3.3.9, we find that the v^{-1} which appears on the right hand side can be ignored (it corresponds to the deletion of a non-critical lateral letter and replacement by v^{-1}), as can the term $v^{-2}\widetilde{t}_1$. The claim follows. □

3.4 Agreement of the f-basis and the canonical basis.

In §3.4, we will often implicitly use the fact that b_i and $\widetilde{t}_j$ commute if and only if b_i and b_j commute if and only if $\widetilde{t}_i$ and $\widetilde{t}_j$ commute.

Lemma 3.4.1. *Let $w \in W_c$, and let $\widehat{f}_w$ be as in Definition 3.3.7. Let f be a monomial in the b_i and $\widetilde{t}_j$ obtained from $\widehat{f}_w$ by changing one critical occurrence of b_p to $\widetilde{t}_p$. Then $\pi_H(f) = \pi_H(\widehat{f}_w)$.*

Proof. It is convenient to split the proof into two cases: $p = 2$ (the case of the bad occurrences of 2) and $p = 2k > 2$. There are no other possibilities (see Proposition 3.1.9).

Suppose $p = 2$ and consider the difference $d = \widehat{f_w} - f$. This is obtained by replacing the deleted occurrence of b_p in $\widehat{f_w}$ by v^{-1}. By Remark 3.1.11, we may apply commutations to d so that it is equal to $xb_3\widetilde{t}_1\widetilde{t}_2\widetilde{t}_1v^{-1}b_3x'$ or its mirror image, where x and x' are monomials in the b_i and $\widetilde{t}_i$. We treat only the first case; the other is the same. Now

$$\widetilde{t}_1\widetilde{t}_2\widetilde{t}_1 = (b_{121} - b_1) - v^{-1}b_2b_1 - v^{-1}b_1b_2 + v^{-2}b_1 + v^{-2}b_2 - v^{-3}$$

which gives

$$\begin{aligned} xb_3\widetilde{t}_1\widetilde{t}_2\widetilde{t}_1v^{-1}b_3x' =& xv^{-1}b_3(b_{121} - b_1)b_3x' \\ & - xb_3v^{-1}b_2b_1v^{-1}b_3x' - xb_3v^{-2}b_1b_2b_3x' \\ & + xb_3v^{-2}b_1v^{-1}b_3x' + xb_3v^{-3}b_2b_3x' - xb_3v^{-4}b_3x'. \end{aligned}$$

The first term in this sum is identically zero, and the others project under π_H to zero by the argument of Lemma 3.3.9. This completes the proof when $p = 2$.

Suppose now that $p = 2k > 2$. In this case, we may apply commutations to $\widehat{f_w}$ and Proposition 3.1.9 (part (i), (ii) or (iii)) to obtain an expression

$$xb_3b_5\cdots b_{2k-1}\widetilde{t}_1\widetilde{t}_2\widetilde{t}_1b_4b_6\cdots b_{2k}b_3b_5\cdots b_{2k-1}x'$$

if we are in case (i) of the proposition,

$$xb_3b_5\cdots b_{2k+1}\widetilde{t}_1\widetilde{t}_2\widetilde{t}_1b_4b_6\cdots b_{2k}b_3b_5\cdots b_{2k-1}x'$$

in case (ii) and

$$xb_3b_5\cdots b_{2k-1}\widetilde{t}_1\widetilde{t}_2\widetilde{t}_1b_4b_6\cdots b_{2k}b_3b_5\cdots b_{2k+1}x'$$

in case (iii). In all cases, the positions of b_3 ensure that the $\widetilde{t}_1$ occurring are lateral occurrences and the $\widetilde{t}_2$ is internal. We may now invoke the argument used to deal with the $p = 2$ case to prove that we may replace the string $\widetilde{t}_1\widetilde{t}_2\widetilde{t}_1$ by $(b_{121} - b_1)$ without changing the result after projection by π_H. Removing the generator b_p

from the expression and replacing it by v^{-1} to produce d now yields, in each case, an expression with

$$b_3 b_5 \cdots b_{2l+1}(b_{121} - b_1) b_4 b_6 \cdots b_{2l} b_3 b_5 \cdots b_{2l+1}$$

somewhere in the middle, where we have $l = k - 1$ in case (i) and $l = k$ in cases (ii) and (iii). This expression is identically zero; this can be seen from the diagram calculus. This proves that $\pi_H(d) = 0$ and completes the proof in the case $p = 2k > 2$. □

Corollary 3.4.2. *Let $w \in W_c$, and let $\widehat{f}_w$ be as in Definition 3.3.7. Let $\widetilde{f}_w$ be the monomial in the b_i and $\widetilde{t}_j$ obtained from $\widehat{f}_w$ by changing all the critical occurrences of b_p to $\widetilde{t}_p$. Then $\pi_H(\widetilde{f}_w) = \pi_H(\widehat{f}_w)$.*

Proof. The proof is by repeated applications of Lemma 3.4.1, starting by replacing the critical occurrences of b_p for the largest possible p. This works because p must be even and, in the proof of Lemma 3.4.1, removal of b_p only involves generators b_q and $\widetilde{t}_q$ with $q < p + 2$. □

We can now prove the main result of §3.

Theorem 3.4.3. *The basis $\{f_w : w \in W_c\}$ is the canonical basis of $TL(H_{n-1})$ in the sense of Theorem 1.2.5.*

Proof. Since f_w is a $\mathbb{Z}$-linear combination of monomials in the b_i, it follows that they are fixed by $\bar{\ }$. It remains to show that $\pi(f_w) = \pi(\widetilde{t}_w)$ for all $w \in W_c$, where π is the projection defined in §1.2.

By Lemma 3.3.10, $\pi_H(f_w) = \pi_H(\widehat{f}_w)$, and by Corollary 3.4.2, $\pi_H(\widehat{f}_w) = \pi_H(\widetilde{f}_w)$. It remains to show that $\pi_H(\widetilde{f}_w) = \pi_H(\widetilde{t}_w)$. This is achieved by the argument in Lemma 3.3.9: we apply Lemma 3.3.3 and then apply Lemma 3.1.3 repeatedly. This is valid since all the problem cases—the internal and critical letters described in Proposition 3.1.9—have been dealt with.

It follows that $\widetilde{t}_w = \sum_{x \in W_c} c_x f_x$, where $c_x \in v^{-1}\mathcal{A}^-$ for $x \neq w$ and $c_w - 1 \in v^{-1}\mathcal{A}^-$. A standard argument (as in the proof of [**6**, Theorem 3.6]) shows that we also have $f_w = \sum_{x \in W_c} c'_x \widetilde{t}_x$, where c'_x satisfies the same properties as c_x above. This shows that $\pi(f_w) = \pi(\widetilde{t}_w)$, which completes the proof. □

4. Properties of the basis in type H.

The aim of §4 is to show that the f-basis constructed in §3 is equal to the diagram basis described in §2. This will prove Theorem 2.1.3.

4.1 Positivity properties for the f-basis.

The aim of §4.1 is to establish a positivity property for the structure constants of $TL(H_{n-1})$ with respect to the f-basis. As in §3, all computations in §4 take place in $TL(H_{n-1})$ over the ring $\mathcal{A}$ unless otherwise stated.

Lemma 4.1.1. *Let $w \in W_c$.*

(i) *Suppose $s \in S$ is such that $ws \notin W_c$. Then there exists a unique $s' \in S$ such that any reduced expression can be parsed in one of the following two ways:*

(a) *$w = w_1 s w_2 s' w_3$, where ss' has order 3, and s commutes with every member of $c(w_2) \cup c(w_3)$;*

(b) *$w = w_1 s w_2 s' w_3 s w_4 s' w_5$, where $\{s, s'\} = \{s_1, s_2\}$, s commutes with every member of $c(w_2) \cup c(w_3) \cup c(w_4) \cup c(w_5)$, and s' commutes with every member of $c(w_3) \cup c(w_4)$.*

(ii) *Suppose $s \in S$ is such that $ws \in W_c$ and the occurrence of s shown is lateral. Let s' be such that $\{s, s'\} = \{s_1, s_2\}$. Then any reduced expression can be parsed in one of the following two ways:*

(a) *$w = w_1 \overline{s} w_2 s' w_3$, where s commutes with every member of $c(w_2) \cup c(w_3)$ and the overscored occurrence of s is not internal;*

(b) *$w = w_1 \overline{s'} w_2 s w_3 s' w_4$, where s commutes with every member of $c(w_3) \cup c(w_4)$, s' commutes with every member of $c(w_2) \cup c(w_3)$ and the overscored occurrence of s' is not internal.*

Proof. The proof of (i) follows exactly the same principles as the corresponding argument in type B. Since the latter argument is presented in full detail in [**7**, Lemma 2.1.2], we omit the proof.

The proof of (ii) follows the same principles. We may apply commutations to the element ws to form a maximal contiguous sequence $ss's$ or $s'ss's$ in which the original occurrence of s appears on the right; no other cases are possible by Lemma 3.2.2 (ii). The first case is described by (a) and the second by (b). (Compare also with Lemma 3.2.5.) □

Lemma 4.1.1 can be refined if we assume the reduced expressions occurring are right justified.

Corollary 4.1.2. *If all reduced expressions occurring in Lemma 4.1.1 are right justified, the situations in parts (i)(b), (ii)(a) and (ii)(b) of that result may be respectively simplified to*

(i) $w = w_1 s s' s s' w_2$, *where* $\{s, s'\} = \{1, 2\}$ *and* s *commutes with every member of* $c(w_2)$;

(ii) $w = w_1 \overline{s} s' w_2$, *where* s *commutes with every member of* $c(w_2)$ *and the overscored occurrence of* s *is not internal;*

(iii) $w = w_1 \overline{s'} s s' w_2$, *where* s *commutes with every member of* $c(w_2)$ *and the overscored occurrence of* s' *is not internal.*

In all three cases, s *and* s' *do not lie in* $c(w_2)$.

Proof. This follows quickly from the definition of "right justified" and Lemma 4.1.1. □

Definition 4.1.3. We set $\mathcal{A}^{\geq 0} := \mathbb{N}[v, v^{-1}]$, where the natural numbers $\mathbb{N}$ are taken to include zero.

If $a, b \in W(X)$ are comparable in the Bruhat–Chevalley order, we define $\max(a, b)$ to be a if $a \geq b$ and b if $b \geq a$.

Lemma 4.1.4. *Let* $w \in W_c$. *Then the structure constants* a_x *in the expression*

$$f_w b_i = \sum_{x \in W_c} a_x f_x$$

satisfy $a_x \in \mathcal{A}^{\geq 0}$. *Furthermore, for all* $a_x \neq 0$, *we have* $x \leq \max(w, ws_i)$ *and* $xs_i < x$, *where* $<$ *is the Bruhat–Chevalley order.*

Proof. We claim that the statement is true for $n = 3$, i.e., $TL(H_2)$. In this case it may be verified that the f-basis is equal to the diagram basis and the other assertions follow from a case-by-case check.

For the case of general n, we proceed by induction on $\ell(w)$. If $\ell(w) < 2$, the statement is obvious. Now consider $f_w b_i$ where $\ell(w) = k > 1$ and the statement is known to be true for $\ell(w) < k$.

There are three cases: the first is $ws_i < w$; the second is $ws_i > w$ and $ws_i \in W_c$; the third is $ws_i > w$ but $ws_i \notin W_c$.

In the first case, w has a reduced expression ending with s_i. It follows from the construction of the f-basis in Definition 3.2.6 that f_w is equal to an element of the form $f'b_i$ for some f'. This means $f_w b_i = (v + v^{-1})f_w$ and the hypotheses are satisfied.

Next, we tackle the second case. There are two subcases according as the rightmost occurrence of s_i is lateral or not. If the occurrence is not lateral, we find that $f_{ws_i} = f_w b_i$, and we are done. If the occurrence is lateral, we are in the situation described in Lemma 4.1.1 (ii). We use the simplification provided by Corollary 4.1.2 and the $n = 3$ case mentioned at the beginning of the proof and set $s := s_i$. In the case of Lemma 4.1.1 (ii) (a), $w = w_1 s s' w_2$ and

$$f_w b_i = f_{w_1}(f_{ss'}b_i)f_{w_2} = f_{w_1}(f_{ss's} + f_s)f_{w_2} = f_{ws} + (f_{w_1 s} f_{w_2}).$$

The term $f_{w_1 s} f_{w_2} = f_{w_1 s} b_{w_2}$ is equal by induction to an $\mathcal{A}^{\geq 0}$-linear combination of basis elements f_x for which x is an ordered subexpression of $w_1 s w_2$, and therefore $x \leq w$. Since s commutes with all elements in $c(w_2)$, the basis elements f_x occurring also satisfy $xs < x$. This completes the first subcase. The second subcase, corresponding to Lemma 4.1.1 (ii) (b), is similar but uses the identity $w = w_1 s' s s' w_2$ and

$$f_w b_i = f_{w_1}(f_{s'ss'}b_i)f_{w_2} = f_{w_1}(f_{s'ss's} + f_{s's})f_{w_2} = f_{ws} + (f_{w_1 ss'} f_{w_2}).$$

The third case is rather similar to the second case, except that there is no term f_{ws}. The identities to use here are $w = w_1 s s' s s' w_2$ and

$$f_w b_i = f_{w_1}(f_{ss'ss'}b_i)f_{w_2} = f_{w_1} f_{ss's} f_{w_2} = f_{w_1 ss's} f_{w_2}.$$

□

Corollary 4.1.5. *Let $w \in W_c$. Then $f_w b_i = (v + v^{-1})f_w$ if and only if $ws_i < w$.*

Proof. If $ws_i < w$ then the proof of Lemma 4.1.4 shows that $f_w b_i = (v + v^{-1})f_w$. For the converse, consider the product $f_w b_i$. In the second case of the proof of Lemma 4.1.4, this product contains f_{ws} with coefficient 1. In the third case of the proof, the product is a linear combination of f_x where x is shorter than w. By the positivity property in Lemma 4.1.4, no cancellation can occur and it is impossible in these cases for $f_w b_i = (v+v^{-1})f_w$. We must therefore be in case (i) of the proof, which gives $ws_i < w$. □

Lemma 4.1.6. *Let $w \in W_c$. Suppose that $ws' < w$ for $s' \in \{s_1, s_2\} = \{s, s'\}$.*

(i) *The structure constants a_x in the expression*

$$f_w(b_s b_{s'} - 1) = \sum_{x \in W_c} a_x f_x$$

satisfy $a_x \in \mathcal{A}^{\geq 0}$.

(ii) *The structure constants a'_x in the expression*

$$f_w(b_s b_{s'} b_s - 2b_s) = \sum_{x \in W_c} a'_x f_x$$

satisfy $a'_x \in \mathcal{A}^{\geq 0}$.

Proof. We proceed by induction on $\ell(w)$, the cases of length 0 and 1 being easy.

Since $w \in W_c$ has a reduced expression ending in s', it has none ending in s, so $ws > w$. There are three cases to consider: first if $ws \in W_c$ and the occurrence of s shown is not lateral; second if $ws \in W_c$ and the occurrence of s shown is lateral and third if $ws \notin W_c$.

For the first case, we first observe that wss' and $wss's$ are also in W_c. This is a consequence of Lemma 4.1.1 (ii). The definition of the f-basis shows that $f_{wss'} = f_w(b_s b_{s'} - 1)$, which proves (i), and that $f_{wss's} = f_w(b_{ss's} - 2b_s)$, which proves (ii).

For the second case, we proceed as in the second case in the proof of Lemma 4.1.4. There are four subcases, corresponding to combinations of either parts (ii) or (iii) of Corollary 4.1.2 and either part (i) or part (ii) of the statement. Let us deal with part (i) of the statement. Here, we have $w = w_1 \overline{s} s' w_2$ or $w = w_1 \overline{s'} s s' w_2$ where $s \notin c(w_2)$ and the overscored letter is not internal. Since we also assume $ws' < w$, we must have $s' \notin c(w_2)$ as well. We now have

$$\begin{aligned} f_w(b_s b_{s'} - 1) &= f_{w_1} f_{ss'} f_{w_2}(b_s b_{s'} - 1) = f_{w_1} f_{ss'}(b_s b_{s'} - 1) f_{w_2} \\ &= f_{w_1}(f_{ss'ss'} + f_{ss'}) f_{w_2} = f_{wss'} + f_{w_1 ss'} f_{w_2}. \end{aligned}$$

Since $f_{w_2} = b_{w_2}$, we are done by Lemma 4.1.4. The other subcase is similar but uses the fact that $f_{s'ss'}(b_s b_{s'} - 1) = f_{s'ss'} + f_{s'}$. The proof for part (ii) follows a similar pattern, again relying on positivity properties in the case $n = 3$: $f_{ss'}(b_s b_{s'} b_s - 2b_s) = f_{ss's}$ and $f_{s'ss'}(b_s b_{s'} b_s - 2b_s) = f_{s's}$.

The third case is reminiscent of the second case. Here, $w = w_1\overline{s}s'ss'w_2$; we need the facts that $f_{ss'ss'}(b_sb_{s'} - 1) = f_{ss'}$ and $f_{ss'ss'}(b_sb_{s'}b_s - 2b_s) = f_s$. □

There are also left-handed versions of the results in lemmas 4.1.4 and 4.1.6. The proofs of these follow by making trivial changes to the arguments, but we present the statements for completeness in the following two lemmas.

Lemma 4.1.7. *Let $w \in W_c$. Then the structure constants a_x in the expression*

$$b_i f_w = \sum_{x \in W_c} a_x f_x$$

satisfy $a_x \in \mathcal{A}^{\geq 0}$. Furthermore, for all $a_x \neq 0$, we have $x \leq \max(w, s_i w)$ and $s_i x < x$, where $<$ is the Bruhat–Chevalley order.

Lemma 4.1.8. *Let $w \in W_c$. Suppose that $s'w < w$ for $s' \in \{s_1, s_2\} = \{s, s'\}$.*

(i) *The structure constants a_x in the expression*

$$(b_{s'}b_s - 1)f_w = \sum_{x \in W_c} a_x f_x$$

satisfy $a_x \in \mathcal{A}^{\geq 0}$.

(ii) *The structure constants a'_x in the expression*

$$(b_sb_{s'}b_s - 2b_s)f_w = \sum_{x \in W_c} a'_x f_x$$

satisfy $a'_x \in \mathcal{A}^{\geq 0}$.

These results have an important consequence which verifies [**7**, Conjecture 1.2.4] for Coxeter systems of type H:

Proposition 4.1.9. *The structure constants for the canonical basis for $TL(H_{n-1})$ lie in $\mathcal{A}^{\geq 0}$.*

Proof. By the construction of the f-basis, we see that f_w is either:

(a) of the form $b_i f_{w'}$ or $f_{w'} b_i$ for some $w' \in W_c$ such that $\ell(w') = \ell(w) - 1$, or

(b) of the form $(b_{s'}b_s - 1)f_{w'}$ or $f_{w'}(b_sb_{s'} - 1)$ for some $w' \in W_c$ such that $s'w' < w'$ (respectively, $w's' < w'$) and $\ell(w') = \ell(w) - 2$ or

(c) of the form $(b_sb_{s'}b_s - 2b_s)f_{w'}$ or $f_{w'}(b_sb_{s'}b_s - 2b_s)$ for some $w' \in W_c$ such that $s'w' < w'$ (respectively, $w's' < w'$) and $\ell(w') = \ell(w) - 3$.

It follows that the multiplication of two basis elements $f_x f_y$ for $x, y \in W_c$ can be broken down into repeated applications of lemmas 4.1.4, 4.1.6, 4.1.7 and 4.1.8, from which the result follows. $\square$

Note that Proposition 4.1.9 is also a consequence of the as yet unproved Theorem 2.1.3 together with [**5**, Proposition 4.1.1].

4.2 Comparison with the diagram basis.

The diagram basis for $TL(H_{n-1})$ can easily be shown to satisfy properties analogous to those described in §4.1 by using results from [**5**].

We denote diagram basis elements by D_w (although we do not specify a bijection between W_c and the set of diagrams).

The following result, which is analogous to lemmas 4.1.4 and 4.1.7, is immediate from the diagram calculus.

Lemma 4.2.1. *The structure constants a_x in the expression*

$$D_w b_i = \sum_{x \in W_c} a_x D_x$$

satisfy $a_x \in \mathcal{A}^{\geq 0}$. Furthermore, for all $a_x \neq 0$, we have $D_x b_i = (v + v^{-1}) D_x$.

An analogous property holds for $b_i D_w$.

The next result is the analogue of lemmas 4.1.6 and 4.1.8. The proof follows easily from the diagram calculus.

Lemma 4.2.2. *Suppose that $D_w b_{s'} = (v + v^{-1}) D_w$ for $s' \in \{s_1, s_2\} = \{s, s'\}$.*

(i) *The structure constants a_x in the expression*

$$D_w(b_s b_{s'} - 1) = \sum_{x \in W_c} a_x D_x$$

satisfy $a_x \in \mathcal{A}^{\geq 0}$.

(ii) *The structure constants a'_x in the expression*

$$D_w(b_s b_{s'} b_s - 2b_s) = \sum_{x \in W_c} a'_x D_x$$

satisfy $a'_x \in \mathcal{A}^{\geq 0}$.

There are also analogous results for $(b_{s'} b_s - 1) D_w$ and $(b_s b_{s'} b_s - 2b_s) D_w$.

The diagram basis has the following key property.

Lemma 4.2.3. *Each diagram basis element D may be obtained from the identity element 1 by repeated application of the following six procedures:*

(i) *left multiplication by b_i;*

(ii) *right multiplication by b_i;*

(iii) *left multiplication of an element D' by $b_{s'}b_s - 1$ where $b_{s'}D' = (v + v^{-1})D'$ and $\{s, s'\} = \{s_1, s_2\}$;*

(iv) *right multiplication of an element D' by $b_s b_{s'} - 1$ where $D'b_{s'} = (v+v^{-1})D'$ and $\{s, s'\} = \{s_1, s_2\}$;*

(v) *left multiplication of an element D' by $b_s b_{s'} b_s - 2b_s$ where $b_{s'}D' = (v + v^{-1})D'$ and $\{s, s'\} = \{s_1, s_2\}$;*

(vi) *right multiplication of an element D' by $b_s b_{s'} b_s - 2b_s$ where $D'b_{s'} = (v+v^{-1})D'$ and $\{s, s'\} = \{s_1, s_2\}$;*

Proof. This is implicit in the proof of [**5**, Proposition 3.2.8]; see also the other results of [**5**, §3.2]. □

We are now ready to prove Theorem 2.1.3.

Proof of Theorem 2.1.3. Combining Lemma 4.2.3 with lemmas 4.1.4, 4.1.6, 4.1.7, 4.1.8 and Corollary 4.1.5 shows that each diagram basis element D is an $\mathcal{A}^{\geq 0}$-linear combination of f-basis elements.

Conversely, combining the construction of the f-basis as in the proof of Proposition 4.1.9 with Corollary 4.1.5 and lemmas 4.2.1 and 4.2.2 shows that each f-basis element is an $\mathcal{A}^{\geq 0}$-linear combination of diagram basis elements.

We observe that a square matrix with entries in $\mathcal{A}^{\geq 0}$ whose inverse has entries in $\mathcal{A}^{\geq 0}$ must be a monomial matrix whose entries are of the form v^k for some $k \in \mathbb{Z}$. Lemma 4.2.3 and the definition of the f-basis show that both the f-basis and the diagram basis are invariant under the map $^-$ of Definition 1.2.4. It follows that all entries in the transition matrices between the f-basis and the diagram basis must be invariant under $^-$, and thus that all entries of the monomial matrix are equal to 1 (i.e., $k = 0$ above). The transition matrices are therefore permutation matrices and the f-basis equals the diagram basis. The proof is completed by Theorem 3.4.3. □

5. Combinatorics in type B.

In this final section, we show how the arguments of §3 and §4 can be adapted to work for type B, and thus to prove Theorem 2.2.5. The argument is essentially a subset of the argument for type H, but we have chosen to present the case of type B separately to avoid making the earlier arguments overly complicated.

In §5, all computations take place in $TL(B_{n-1})$ over the ring $\mathcal{A}$ unless otherwise stated.

5.1 Basic properties of B-canonical diagrams.

Lemma 5.1.1. *The set C_n of Theorem 2.2.5 is linearly independent over $\mathcal{A}$, and its structure constants lie in $\mathcal{A}^{\geq 0}$.*

Proof. The fact that the set is linearly independent follows from comparing definitions 2.2.2 and 2.2.4.

It is obvious from the relations in figures 4 and 5 that the structure constants for this set lie in $\mathcal{A}[\frac{1}{2}]$.

The proof is a case by case check, multiplying elements of C_n of various types (C1, C1′, C2 as in Definition 2.2.4) together. The two difficult cases are (a) removal of a (single) loop decorated by a circle and (b) what to do when a diagram emerges with a (single instance of a) circular decoration where a square is required. In each of these cases, there is a spare factor of 2 available; this deals with case (a) immediately. For case (b), we apply the first relation in Figure 5 to reexpress the diagram as a linear combination of two elements of C_n, each with coefficient 1. □

The next problem to resolve is the issue of which $\mathcal{A}$-form of the algebra $TL(B_{n-1})$ to use. (The one given in [**4**] is the wrong one for our purposes.)

We start by defining an injective map from the set of B-canonical diagrams (see Definition 2.2.4) to the set of H-admissible diagrams (see Definition 2.1.1).

Definition 5.1.2. Let D be a B-canonical diagram, and let $\lambda D \in C_n$ (so that $\lambda = 1$ or 2). Then the linear map $\iota : TL(B_{n-1}) \longrightarrow TL(H_{n-1})$ is defined to take λD to the H-admissible diagram obtained by replacing all the decorations in D (whether circular or square) by circular decorations.

Lemma 5.1.3. *The map ι is well-defined and injective.*

Proof. This follows from examination of definitions 2.1.1 and 2.2.4. □

The next part of the argument is an adaptation of the argument in [**5**, §3.2] to type B. The proofs of the following four lemmas, in which $D \in C_n$, are immediate.

Lemma 5.1.4. *Assume D has a propagating edge, E, connecting node p_1 in the north face to node p_2 in the south face.*

If nodes $p_1 + 1$ and $p_1 + 2$ in the north face are connected by a (necessarily undecorated) edge E', then $b_{p_1} D$ is the element of C_n obtained by removing E', disconnecting E from the north face and reconnecting it to node $p_1 + 2$ in the north face, and installing a new undecorated edge between points p_1 and $p_1 + 1$ in the north face. The edge corresponding to E retains its original decoration status.

Furthermore, $\iota(b_{p_1} D) = b_{p_1} \iota(D)$ in $TL(H_{n-1})$.

Lemma 5.1.5. *Assume that $i > 1$, and that in the north face of D, nodes i and $i+1$ are connected by a (square-)decorated edge e_1, and nodes $i+2$ and $i+3$ are connected by an undecorated edge, e_2. Then $b_i b_{i+1} D$ is the element of C_n obtained from D by exchanging e_1 and e_2. This procedure has an inverse, since $D = b_{i+2} b_{i+1} b_i b_{i+1} D$.*

Furthermore, $\iota(b_i b_{i+1} D) = b_i b_{i+1} \iota(D)$ in $TL(H_{n-1})$.

Lemma 5.1.6. *Assume that in the north face of D, nodes 1 and 2 are connected by a decorated edge (necessarily decorated by a circle), and nodes 3 and 4 are connected by an undecorated edge. Then $(b_1 b_2 - 1)D \in C_n$ is the element obtained from D by decorating the edge connecting nodes 3 and 4 with a square.*

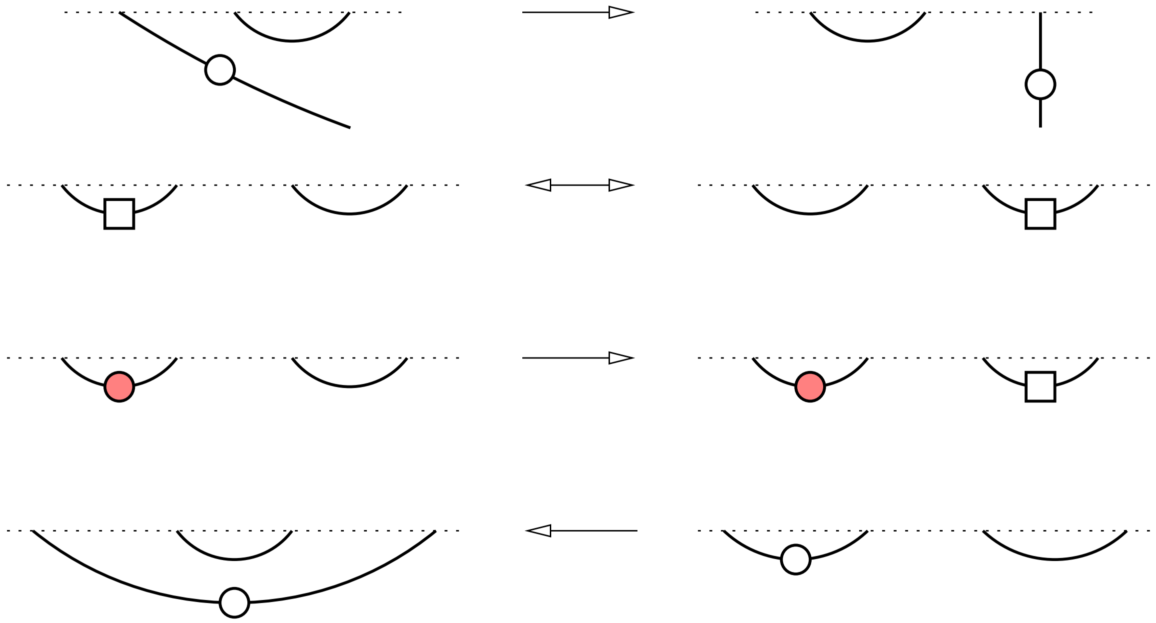
Furthermore, $\iota((b_1 b_2 - 1)D) = (b_1 b_2 - 1)\iota(D)$ in $TL(H_{n-1})$.

Lemma 5.1.7. *Assume that in the north face of D, nodes i and $i+1$ are connected by an undecorated edge, e_1, and nodes $j < i$ and $k > i+1$ are connected by an edge, e_2. Assume also that j and k are chosen such that $|k - j|$ is minimal. Then D is of the form $b_i D'$, where D' is an element of C_n which is the same as D except as regards the edges connected to nodes $j, i, i+1, k$ in the north face. Nodes j and i in D' are connected to each other by an edge with the same decoration as e_2, and nodes $i+1$ and k are connected to each other by an undecorated edge.*

Furthermore, $\iota(b_i D') = b_i \iota(D')$ in $TL(H_{n-1})$.

These results may be visualized as in Figure 13, in which an unshaded circular decoration denotes an optional decoration of indeterminate type, thus depicting the concept of "original decoration status" in Lemma 5.1.4. There are also right-handed versions of the results, which correspond to top-bottom reflected versions of Figure 13.

FIGURE 13. Lemmas 5.1.4–5.1.7



The following property of the diagram calculus is analogous to that described in Lemma 4.2.3 for type H.

Lemma 5.1.8. *Each element of the set C_n of Theorem 2.2.5 may be obtained from the identity element 1 by repeated application of the following four procedures:*

(i) *left multiplication by b_i;*

(ii) *right multiplication by b_i;*

(iii) *left multiplication of an element D' by $b_{s'}b_s - 1$ where $b_{s'}D' = (v + v^{-1})D'$ and $\{s, s'\} = \{s_1, s_2\}$;*

(iv) *right multiplication of an element D' by $b_s b_{s'} - 1$ where $D'b_{s'} = (v + v^{-1})D'$ and $\{s, s'\} = \{s_1, s_2\}$;*

Proof. The argument is similar to the argument in [**5**, §3.2] for type H. We first note that the set of Theorem 2.2.5 consists, in type B_2, of the elements

$$1, b_1, b_2, b_1b_2, b_2b_1, (b_1b_2 - 1)b_1, (b_2b_1 - 1)b_2,$$

which agrees with the hypotheses.

The general case uses lemmas 5.1.4–5.1.7. The results about ι mean that we may copy the argument of [**5**, Proposition 3.2.8], from which the conclusion follows. $\square$

Proposition 5.1.9. *The $\mathcal{A}$-spans of the following sets are equivalent:*

(i) $\{t_w : w \in W_c\}$,

(ii) $\{b_w : w \in W_c\}$,

(iii) *the set C_n in the statement of Theorem 2.2.5.*

Proof. The equivalence of the first two follows from the fact that they are both $\mathcal{A}$-bases for $TL(X)$; see the remarks in Definition 1.2.4. We prove that the sets in (ii) or (iii) have the same $\mathcal{A}$-span, implicitly using the fact that the set in (i) spans an $\mathcal{A}$-algebra.

First consider b_w for $w \in W_c$. Since $b_i \in C_n$, repeated applications of Lemma 5.1.1 show that b_w lies in the $\mathcal{A}$-span of the set in (iii). Conversely, Lemma 5.1.8 shows that each element of the set in (iii) is a polynomial (over $\mathbb{Z}$) in the elements b_i, which establishes the reverse inclusion. □

5.2 Main results in type B.

Since the set C_n of Theorem 2.2.5 produces the correct integral form, the arguments of §3 and §4 may be easily adapted to type B. In general, these arguments are subsets of the arguments in type H: the difference is that strings $s_1s_2s_1s_2$ and $s_2s_1s_2s_1$ are not allowed in elements of W_c, which eliminates several of the most complicated cases. For reasons of space, we describe only the necessary changes to the proof, together with statements of the main intermediate results.

Lemmas 3.1.1–3.1.3 have direct analogues for type B, replacing the diagram basis in type H with the set C_n. The set C_n leads to the definition of a projection π_B analogous to π_H. The definitions of internal and lateral are the same as in type H. Cases (ii) and (iii) in Proposition 3.1.9 cannot actually occur in practice, but this is neither immediate nor necessary to get the argument to work.

The definition of the f-basis in type B is formally identical to type H in §3.2, so we end up with a set which is formally the same as a subset of the f-basis in type H.

The results of §3.3 adapt readily to type B, the main difference being that the case dealt with by Lemma 3.3.6 (ii) is not needed.

Copying the results of §3.4, we obtain the following analogue of Theorem 3.4.3.

Theorem 5.2.1. *The basis $\{f_w : w \in W_c\}$ is the canonical basis of $TL(B_{n-1})$ in the sense of Theorem 1.2.5.*

The results of §4 also adapt easily; the main changes to make are the omissions of some of the cases.

For Lemma 4.1.1, we replace (i) (b) with its counterpart for type B given in [**7**, Lemma 2.1.2 (ii)]. The case in Lemma 4.1.1 (ii) (b) cannot occur and can be ignored; the same goes for the situation in Corollary 4.1.2 (iii). Similar remarks hold for Lemma 4.1.6 (ii) and Lemma 4.1.8 (ii). This allows us to prove an analogue of Proposition 4.1.9 which verifies [**7**, Conjecture 1.2.4] for Coxeter systems of type B.

Proposition 5.2.2. *The structure constants for the canonical basis for $TL(B_{n-1})$ lie in $\mathcal{A}^{\geq 0}$.*

The cases to be ignored in §4.2 are Lemma 4.2.2 (ii) and Lemma 4.2.3 (v) and (vi).

Theorem 2.2.5 follows. □

5.3 Concluding remarks.

It is natural to wonder whether short proofs of Theorem 2.1.3 and Theorem 2.2.5 exist. It turns out (although we will not pursue this here) that these results are closely related to the following hypothesis.

Hypothesis 5.3.1. *Consider a generalized Temperley–Lieb algebra, $TL(X)$ with t-basis $\{\widetilde{t}_w : w \in W_c\}$. Then there is a symmetric, anti-associative, nondegenerate $\mathcal{A}$-bilinear form $\langle , \rangle$ on $TL(X)$ with respect to which*

$$\langle \widetilde{t}_w, \widetilde{t}_x \rangle = \delta_{w,x} \mod v^{-1}\mathcal{A}^-,$$

where δ is the Kronecker delta.

By "anti-associative" above, we mean that $\langle \widetilde{t}_s \widetilde{t}_w, \widetilde{t}_x \rangle = \langle \widetilde{t}_w, \widetilde{t}_s \widetilde{t}_x \rangle$.

Note that there is an analogous hypothesis for Hecke algebras $\mathcal{H}(X)$, but that the existence of such a form is well-known and almost trivial (set $\langle T_w, T_x \rangle := \delta_{w,x} q^{\ell(w)}$). It is not hard to establish Hypothesis 5.3.1 in type H from Theorem 2.1.3 and in type B from Theorem 2.2.5, but proving it directly is curiously difficult. Conversely, given Hypothesis 5.3.1, we may characterize the canonical basis up to sign using the bilinear form as in [**11**, §14.2] and then show, using a short argument, that the relevant diagram basis satisfies this characterization and some mild positivity assumptions. We will return to the consideration of Hypothesis 5.3.1 in a subsequent paper.

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