STAR REDUCIBLE COXETER GROUPS

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Abstract. We define "star reducible" Coxeter groups to be those Coxeter groups for which every fully commutative element (in the sense of Stembridge) is equivalent to a product of commuting generators by a sequence of length-decreasing star operations (in the sense of Lusztig). We show that the Kazhdan–Lusztig bases of these groups have a nice projection property to the Temperley–Lieb type quotient, and furthermore that the images of the basis elements C'_w (for fully commutative w) in the quotient have structure constants in $\mathbb{Z}^{\geq 0}[v, v^{-1}]$. We also classify the star reducible Coxeter groups and show that they form nine infinite families with two exceptional cases.

Introduction

Let (W, S) be a Coxeter group, with finite generating set S. Stembridge [**19**] introduced the set W_c of fully commutative elements of W as those for which any two reduced expressions in the generators are equivalent via iterated application of short braid relations, that is, relations of the form $ss' = s's$, where $s, s' \in S$. For example, if w is a product of commuting generators from S, then w is fully commutative.

If $I = \{s, s'\} \subseteq S$ is a pair of noncommuting Coxeter generators, then I induces four partially defined maps from W to itself, known as star operations. A star

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operation, when it is defined, respects the partition $W = W_c \dot{\cup} (W \backslash W_c)$ of the Coxeter group, and increases or decreases the length of the element to which it is applied by 1.

In this paper we will analyse the situation where every fully commutative element can be reduced to a product of commuting generators from S by iterated application of length-decreasing star operations; this property is called "Property F" in [**12**], as it is essentially the same as Fan's notion of cancellability in [**5**]. Groups with this property are the eponymous "star reducible Coxeter groups", and they include the finite Coxeter groups as a subclass.

We shall show (Theorem 4.1) that arbitrary elements of star reducible Coxeter groups have reduced expressions of a particularly nice type, which allows us to prove (Theorem 4.3) a strong form of a certain conjectured projection property (in the sense of [**14**, **18**]) for the associated Kazhdan–Lusztig basis $\{C'_w : w \in W\}$. This has some strong consequences (Theorem 4.6) for the Kazhdan–Lusztig type basis $\{c_w : w \in W_c\}$ introduced by J. Losonczy and the author for a Temperley–Lieb type quotient of the Hecke algebra $\mathcal{H}$ associated to W. In the star reducible case, this basis turns out simply to be the projection of the Kazhdan–Lusztig basis elements $\{C'_w : w \in W_c\}$. Furthermore, there is a simple inductive construction for the c_w, and the c-basis can be shown to have nonnegative structure constants, that is, structure constants that are Laurent polynomials with nonnegative coefficients. One of the reasons this is interesting is that in many cases (see [**12**, §6] and [**15**, Theorem 2.2.3, §3.1]), these structure constants are also structure constants for the Kazhdan–Lusztig basis, whose positivity is generally very difficult to prove.

Finally (Theorem 6.3), we classify all star reducible Coxeter groups for which S is a finite set. This class of groups contains the seven infinite families of groups (A, B, D, E, F, H and I) for which W_c is finite, which were classified independently by Graham [**8**] and Stembridge [**19**], as well as three other infinite families (one of which subsumes type I) and two exceptional cases.

Combining the main result of this paper (Theorem 4.6) with the classification of star reducible Coxeter groups (Theorem 6.3), one obtains an extensive class of examples of situations where the projection of the Kazhdan–Lusztig basis elements C'_w (for fully commutative w) to the Temperley–Lieb quotient have positive structure constants. These quotients are useful because they provide combinatorially tractable models for Kazhdan–Lusztig theory that are useful for formulating and checking conjectures, and in a future paper we plan to explain the application of the quotient algebras to the representation theory of the corresponding Lie algebras. Our results here also provide unifying conceptual proofs for various results already in the literature.

1. Preliminaries

Let X be a Coxeter graph, of arbitrary type, and let $W = W(X)$ be the associated Coxeter group with distinguished (finite) set of generating involutions $S(X)$. (The reader is referred to [**1**] or [**16**] for details of the theory of Coxeter groups.) In other words, $W = W(X)$ is given by the presentation

$$W = \langle S(X) \mid (st)^{m(s,t)} = 1 \text{ for } m(s,t) < \infty \rangle,$$

where $m(s, s) = 1$ and $m(s, t) = m(t, s)$. It turns out that the elements of $S = S(X)$ are distinct as group elements, and that $m(s, t)$ is the order of st.

Denote by S^* the free monoid on $S = S(X)$. We call the elements of S *letters* and those of S^* *words*. The *length* of a word is the number of factors required to write the word as a product of letters. Let $\phi : S^* \longrightarrow W$ be the surjective morphism of monoid structures satisfying $\phi(i) = s_i$ for all $i \in S$. A word $\mathbf{i} \in S^*$ is said to *represent* its image $w = \phi(\mathbf{i}) \in W$; furthermore, if the length of $\mathbf{i}$ is minimal among the lengths of all the words that represent w, then we call $\mathbf{i}$ a *reduced expression* for w. The *length* of w, denoted by $\ell(w)$, is then equal to the length of $\mathbf{i}$. A product $w_1 w_2 \cdots w_n$ of elements $w_i \in W$ is called *reduced* if $\ell(w_1 w_2 \cdots w_n) = \sum_i \ell(w_i)$. We write

$$\mathcal{L}(w) = \{s \in S : \ell(sw) < \ell(w)\}$$

and

$$\mathcal{R}(w) = \{s \in S : \ell(ws) < \ell(w)\}.$$

The set $\mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) is called the *left* (respectively, *right*) *descent set* of w.

The *commutation monoid* $\mathrm{Co}(X, S)$ is the quotient of the free monoid S^* by the congruence $\equiv$ generated by the commutation relations:

$$st \equiv ts \text{ for all } s, t \in S \text{ with } \phi(s)\phi(t) = \phi(t)\phi(s);$$

note that, as a monoid, W is a quotient of $\mathrm{Co}(X, S)$.

The elements of $\mathrm{Co}(X, S)$, which computer scientists call *traces* [**3**], have the following normal form, often called the Cartier–Foata normal form (see [**2**]).

Theorem 1.1 (Cartier–Foata normal form). *Let* $\mathbf{s}$ *be an element of the commutation monoid* $\mathrm{Co}(X, S)$*. Then* $\mathbf{s}$ *has a unique factorization in* $\mathrm{Co}(X, S)$ *of the form*

$$\mathbf{s} = \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_p$$

such that each $\mathbf{s}_i$ *is a product of distinct commuting elements of* S*, and such that for each* $1 \leq j < p$ *and each generator* $t \in S$ *occurring in* $\mathbf{s}_{j+1}$*, there is a generator* $s \in S$ *occurring in* $\mathbf{s}_j$ *such that* $st \neq ts$*.* $\square$

Remark 1.2. The Cartier–Foata normal form may be defined inductively, as follows. If we define $\mathcal{L}(\mathbf{s})$ to be the set of possible first letters in all the words $\mathbf{s}'$ for which $\mathbf{s}' \equiv \mathbf{s}$ in $\mathrm{Co}(X, S)$, then $\mathbf{s}_1$ is just the product of the elements in $\mathcal{L}(\mathbf{s})$. Since $\mathrm{Co}(X, S)$ is a cancellative monoid, there is a unique element $\mathbf{s}' \in \mathrm{Co}(X, S)$ with $\mathbf{s} = \mathbf{s}_1 \mathbf{s}'$. If

$$\mathbf{s}' = \mathbf{s}_2 \cdots \mathbf{s}_p$$

is the Cartier–Foata normal form of $\mathbf{s}'$, then

$$\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_p$$

is the Cartier–Foata normal form of $\mathbf{s}$.

Denote by $\mathcal{H} = \mathcal{H}(X)$ the Hecke algebra associated to W. This is a $\mathbb{Z}[q, q^{-1}]$-algebra with a basis consisting of (invertible) elements T_w, with w ranging over W, satisfying

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where ℓ is the length function on the Coxeter group W, $w \in W$, and $s \in S$.

For many applications it is convenient to introduce an $\mathcal{A}$-form of $\mathcal{H}$, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and $v^2 = q$, and to define a scaled version of the T-basis, $\{\widetilde{T}_w : w \in W\}$, where $\widetilde{T}_w := v^{-\ell(w)}T_w$. Unless otherwise stated, we will use the $\mathcal{A}$-form of $\mathcal{H}$ from now on, and we will denote the $\mathbb{Z}[q, q^{-1}]$-form by $\mathcal{H}_q$. We will write $\mathcal{A}^+$ and $\mathcal{A}^-$ for $\mathbb{Z}[v]$ and $\mathbb{Z}[v^{-1}]$, respectively, and we denote the $\mathbb{Z}$-linear ring homomorphism $\mathcal{A} \longrightarrow \mathcal{A}$ exchanging v and v^{-1} by $^-$. We can extend $^-$ to a ring automorphism of $\mathcal{H}$ (as in [**7**, Theorem 11.1.10]) by the condition that

$$\overline{\sum_{w \in W} a_w \widetilde{T}_w} := \sum_{w \in W} \overline{a_w} \widetilde{T}^{-1}_{w^{-1}},$$

where the a_w are elements of $\mathcal{A}$.

In [**17**], Kazhdan and Lusztig proved the following

Theorem 1.3. (Kazhdan, Lusztig). *For each $w \in W$, there exists a unique $C'_w \in \mathcal{H}$ such that both $\overline{C'_w} = C'_w$ and*

$$C'_w = \widetilde{T}_w + \sum_{y<w} a_y \widetilde{T}_y,$$

where $<$ is the Bruhat order on W and $a_y \in v^{-1}\mathcal{A}^-$. The set $\{C'_w : w \in W\}$ forms an $\mathcal{A}$-basis for $\mathcal{H}$. □

Following [**7**, §11.1], we denote the coefficient of $\widetilde{T}_y$ in C'_w by $P^*_{y,w}$. The *Kazhdan–Lusztig polynomial* $P_{y,w}$ is then given by $v^{\ell(w)-\ell(y)}P^*_{y,w}$.

Let $J(X)$ be the two-sided ideal of $\mathcal{H}$ generated by the elements

$$\sum_{w \in \langle s, s' \rangle} T_w,$$

where (s, s') runs over all pairs of elements of S that correspond to adjacent nodes in the Coxeter graph, and $\langle s, s' \rangle$ is the parabolic subgroup generated by s and s'. (If the nodes corresponding to (s, s') are connected by a bond of infinite strength, then we omit the corresponding relation.)

Following Graham [**8**, Definition 6.1], we define the *generalized Temperley–Lieb algebra* $TL(X)$ to be the quotient $\mathcal{A}$-algebra $\mathcal{H}(X)/J(X)$. We denote the corresponding epimorphism of algebras by $\theta : \mathcal{H}(X) \longrightarrow TL(X)$. Since the generators of $J(X)$ lie in $\mathcal{H}_q(X)$, we also obtain a $\mathbb{Z}[q, q^{-1}]$-form $TL_q(X)$, of $TL(X)$. Let t_w (respectively, $\widetilde{t}_w$) denote the image in $TL(X)$ of the basis element T_w (respectively, $\widetilde{T}_w$) of $\mathcal{H}$.

Call an element $w \in W$ *complex* if it can be written as a reduced product $x_1 w_{ss'} x_2$, where $x_1, x_2 \in W$ and $w_{ss'}$ is the longest element of some rank 2 parabolic subgroup $\langle s, s' \rangle$ such that s and s' correspond to adjacent nodes in the Coxeter graph. An element $w \in W$ is said to be *weakly complex* if it is complex and of the form $w = su$, where u is not complex and $s \in S$. In this case, we must have $su > u$.

Denote by $W_c(X)$ the set of all elements of W that are not complex. The elements of W_c are the *fully commutative* elements of [**19**]; they are characterized by the property that any two of their reduced expressions may be obtained from each other by repeated commutation of adjacent generators; in other words, all reduced expressions are equal as elements of $\mathrm{Co}(X, S)$. Each reduced expression for w has a Cartier–Foata normal form, by considering it as an element of $\mathrm{Co}(X, S)$, and this normal form is an invariant of w if and only if w is fully commutative.

We define the $\mathcal{A}^-$-submodule $\mathcal{L}$ of $TL(X)$ to be that generated by $\{\widetilde{t}_w : w \in W_c\}$. We define $\pi : \mathcal{L} \longrightarrow \mathcal{L}/v^{-1}\mathcal{L}$ to be the canonical $\mathbb{Z}$-linear projection.

By [**13**, Lemma 1.4], the ideal $J(X)$ is fixed by $\bar{\ }$, so $\bar{\ }$ induces an involution on

$TL(X)$, which we also denote by $^-$.

The following result is an analogue of Theorem 1.3 for the quotient algebra.

Theorem 1.4.

(i) *The set* $\{t_w : w \in W_c\}$ *is a* $\mathbb{Z}[q, q^{-1}]$*-basis for* $TL_q(X)$*. The set* $\{\widetilde{t}_w : w \in W_c\}$ *is an* $\mathcal{A}$*-basis for* $TL(X)$*, and an* $\mathcal{A}^-$*-basis for* $\mathcal{L}$*.*

(ii) *For each* $w \in W_c$*, there exists a unique* $c_w \in TL(X)$ *such that both* $\overline{c_w} = c_w$ *and* $\pi(c_w) = \pi(\widetilde{t}_w)$*. Furthermore, we have*

$$c_w = \widetilde{t}_w + \sum_{\substack{y < w \\ y \in W_c}} a_y \widetilde{t}_y,$$

where $<$ *is the Bruhat order on* W*, and* $a_y \in \mathcal{A}^-$ *for all* y*.*

(iii) *The set* $\{c_w : w \in W_c\}$ *forms an* $\mathcal{A}$*-basis for* $TL(X)$ *and an* $\mathcal{A}^-$*-basis for* $\mathcal{L}$*.*

(iv) *If* $x \in \mathcal{L}$ *and* $\bar{x} = x$*, then* x *is a* $\mathbb{Z}$*-linear combination of the* c_w*.*

Proof. This is a subset of [**12**, Theorem 2.1]. (Note that (i) is due to Graham [**8**, Theorem 6.2], and (ii) and (iii) are essentially due to J. Losonczy and the author [**13**, Theorem 2.3].) $\square$

Let W be any Coxeter group and let $I = \{s, t\} \subseteq S$ be a pair of noncommuting generators whose product has order m (where $m = \infty$ is allowed). Let W^I denote the set of all $w \in W$ satisfying $\mathcal{L}(w) \cap I = \emptyset$. Standard properties of Coxeter groups [**16**, §5.12] show that any element $w \in W$ may be uniquely written as $w = w_I w^I$, where $w_I \in W_I = \langle s, t \rangle$ and $\ell(w) = \ell(w_I) + \ell(w^I)$. There are four possibilities for elements $w \in W$:

(i) w is the shortest element in the coset $W_I w$, so $w_I = 1$ and $w \in W^I$;

(ii) w is the longest element in the coset $W_I w$, so w_I is the longest element of W_I (which can only happen if W_I is finite);

(iii) w is one of the $(m-1)$ elements sw^I, tsw^I, $stsw^I, \ldots$;

(iv) w is one of the $(m-1)$ elements tw^I, stw^I, $tstw^I, \ldots$.

The sequences appearing in (iii) and (iv) are called *(left) $\{s,t\}$-strings*, or *strings* if the context is clear. If x and y are two elements of an $\{s,t\}$-string such that $\ell(x) = \ell(y) - 1$, we call the pair $\{x, y\}$ *left $\{s,t\}$-adjacent*, and we say that y is *left star reducible to x*.

The above concepts all have right-handed counterparts, leading to the notion of *right $\{s,t\}$-adjacent* and *right star reducible* pairs of elements, and coset decompositions $({}^I w)({}_I w)$.

If there is a (possibly trivial) sequence

$$x = w_0, w_1, \dots, w_k = y$$

where, for each $0 \leq i < k$, w_{i+1} is left star reducible or right star reducible to w_i with respect to some pair $\{s_i, t_i\}$, we say that y is *star reducible to x*. Because star reducibility decreases length, it is clear that this defines a partial order on W.

If w is an element of an $\{s,t\}$-string, S_w, we have $\{\ell(sw), \ell(tw)\} = \{\ell(w) - 1, \ell(w) + 1\}$; let us assume without loss of generality that sw is longer than w and tw is shorter. If sw is an element of S_w, we define ${}^*w = sw$; if not, ${}^*w$ is undefined. If tw is an element of S_w, we define ${}_*w = tw$; if not, ${}_*w$ is undefined.

There are also obvious right handed analogues to the above concepts, so the symbols w^* and w_* may be used with the analogous meanings.

Example 1.5. In the Coxeter group of type B_2 with $w = ts$, we have

$${}_*w = s, \ {}^*w = sts, \ w_* = t \text{ and } w^* = tst.$$

If $x = sts$ then ${}^*x$ and x^* are undefined; if $x = t$ then ${}_*x$ and x_* are undefined.

Definition 1.6. We say that a Coxeter group $W(X)$, or its Coxeter graph X, is *star reducible* if every element of W_c is star reducible to a product of commuting generators from S.

2. Acyclic monomials

In order to derive some of the results in this paper, and §2 in particular, we will need to use the author's theory of acyclic heaps [**10**, **11**]. Heaps, as introduced by Viennot in [**21**], are certain combinatorial structures associated to elements of $\mathrm{Co}(X, S)$; they are known as "dependence graphs" in the computer science literature [**3**]. However, in order to keep the paper as accessible as possible, we will avoid mention of heaps and work directly with monomials, or traces. All Coxeter groups in §2 will be star reducible.

Theorem 2.1. *Let (W, S) be a star reducible Coxeter group. There is a unique function $h : \mathrm{Co}(X, S) \longrightarrow \mathbb{Z}^{\geq 0}$ with the following properties.*

(i) *If $\mathbf{u} \in \mathrm{Co}(X, S)$ and $s, t \in S$ are noncommuting generators, then $h(st\mathbf{u}) = h(t\mathbf{u})$ and $h(\mathbf{u}ts) = h(\mathbf{u}t)$.*

(ii) *If $\mathbf{u} \in \mathrm{Co}(X, S)$ is represented by a monomial*

$$s_1 s_2 \cdots s_r$$

that is a reduced expression for some $w \in W_c$, then $h(\mathbf{u}) = 0$.

(iii) *If $\mathbf{u} = \mathbf{u}_1 ss \mathbf{u}_2$ for some generator $s \in S$, and $\mathbf{u}' = \mathbf{u}_1 s \mathbf{u}_2$, then $h(\mathbf{u}) = h(\mathbf{u}') + 1$.*

(iv) *If $\mathbf{u} = \mathbf{u}_1 sts \mathbf{u}_2$ for some noncommuting generators $s, t \in S$, and $\mathbf{u}' = \mathbf{u}_1 s \mathbf{u}_2$, then $h(\mathbf{u}) = h(\mathbf{u}')$.*

(v) *If $\mathbf{u} = \mathbf{u}_1 s \mathbf{u}_2$ for some generator $s \in S$, and $\mathbf{u}' = \mathbf{u}_1 \mathbf{u}_2$, then $|h(\mathbf{u}) - h(\mathbf{u}')| \leq 1$.*

Proof. Let k be a field.

According to [**21**, Proposition 3.4], elements $\mathbf{u}$ of $\mathrm{Co}(X, S)$ are in bijection with certain heaps $[E, \leq, \varepsilon]$ (see [**10**], and [**10**, Proposition 3.1.4] in particular, for more details on these concepts and the notation). Let $h(\mathbf{u}) = \dim H_1(E, k)$; it will turn out that the definition is independent of k.

Part (i) follows from the proof of the inductive step in [**10**, Proposition 2.2.3].

Since W is star reducible, it follows from using (i) repeatedly that (ii) is true if and only if it is true when $\mathbf{u}$ is a product of distinct commuting generators. In this case, the claim follows from the proof of the base case of the induction in [**10**, Proposition 2.2.3].

Part (iii) is a restatement of [**10**, Lemma 2.3.4], part (iv) is a restatement of [**10**, Lemma 2.3.5], and part (v) is a restatement of [**10**, Theorem 2.1.1] (star reducibility plays no role in these proofs).

It follows from [**1**, Theorem 3.3.1 (i)] that the elements of $\mathrm{Co}(X, S)$ corresponding to reduced expressions of some $w \in W$ are precisely those that have no monomial representative of the form $\mathbf{u}_1 ss \mathbf{u}_3$, where $s \in S$, and no monomial representative $\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3$ where $\mathbf{u}_2$ is an alternating product of $m(s,t) > 2$ occurrences of s and t. It follows from this that any element of $\mathrm{Co}(X, S)$ can be transformed into an element of $\mathrm{Co}(X, S)$ corresponding to a reduced expression for some $w \in W_c$ by repeatedly applying transformations of the form $ss \mapsto s$ or $sts \mapsto s$, as used in parts (iii) and (iv). Applying (ii), we see there is at most one function h satisfying (ii), (iii) and (iv). This proves uniqueness of h and also shows that the definition is independent of the choice of field k. $\square$

Definition 2.2. In the set-up of Theorem 2.1, an element $\mathbf{u}$ of $\mathrm{Co}(X, S)$ (and, by extension, an element of S^* representing $\mathbf{u}$) is called an *acyclic monomial* if $h(\mathbf{u}) = 0$. (The acyclic monomials are those that correspond to the acyclic heaps of [**10**, **11**].)

For our purposes in this paper, it is convenient to work with another basis of $TL(X)$, namely the monomial basis. Although the fact that this is a basis is well-known, we provide a proof since there does not seem to be an easily available general proof in the literature.

Definition 2.3. Let W be a Coxeter group and let $w \in W_c$ be a fully commutative element. Let

$$w = s_1 s_2 \cdots s_r$$

be a reduced expression for w. For each $s \in S$, let $b_s = v^{-1}\widetilde{t}_1 + \widetilde{t}_s$, then define

$b_w \in TL(X)$ by

$$b_w := b_{s_1} b_{s_2} \cdots b_{s_r}.$$

Note that the element b_w is well-defined precisely because any two reduced expressions for w are commutation equivalent.

Proposition 2.4. *The set $\{b_w : w \in W_c\}$ is a free $\mathcal{A}$-basis for $TL(X)$, and $\overline{b_w} = b_w$ for all $w \in W_c$.*

Proof. The second assertion follows from the fact that $^-$ is a ring endomorphism of $TL(X)$ that fixes the generators $b_s = c_s (s \in S)$.

To prove the first assertion, first observe that by definition of the ideal $J(X)$, we have the relation

$$\widetilde{t}_{w_{ss'}} = - \sum_{w \in \langle s, s' \rangle, w < w_{ss'}} v^{\ell(w) - \ell(w_{ss'})} \widetilde{t}_w. \tag{1}$$

in $TL(X)$, where $w_{ss'}$ is the longest element in the parabolic subgroup $\langle s, s' \rangle$ of W. This has the consequence that any monomial

$$\widetilde{t}_{s_1} \widetilde{t}_{s_2} \cdots \widetilde{t}_{s_k},$$

where all $s_i \in S$, can be expressed as a linear combination of basis elements $\widetilde{t}_x$ for which $\ell(x) \leq k$. Now let $x \in W_c$ and let $s_1 s_2 \cdots s_r$ be a reduced expression for x. Since

$$b_x = b_{s_1} b_{s_2} \cdots b_{s_r},$$

we have

$$b_x = (v^{-1} \widetilde{t}_1 + \widetilde{t}_{s_1})(v^{-1} \widetilde{t}_1 + \widetilde{t}_{s_2}) \cdots (v^{-1} \widetilde{t}_1 + \widetilde{t}_{s_r}).$$

Expanding the parentheses and using equation (1), we see that

$$b_x = \widetilde{t}_x + \sum_{\substack{y \in W_c \\ \ell(y) < \ell(x)}} a_y \widetilde{t}_y$$

for some coefficients $a_y \in \mathcal{A}$. It is now clear that the set in the statement is a basis, and that the change of basis matrix from the $\widetilde{t}$-basis to the b-basis is unitriangular. $\square$

It will be convenient to have a presentation of $TL(X)$ in terms of the generators b_s; compare with [**8**, Proposition 9.5].

Definition 2.5. We define the Chebyshev polynomials of the second kind to be the elements of $\mathbb{Z}[x]$ given by the conditions $P_0(x) = 1$, $P_1(x) = x$ and

$$P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$$

for $n \geq 2$. If $f(x) \in \mathbb{Z}[x]$, we define $f_b^{s,t}(x)$ to be the element of $TL(X)$ given by the linear extension of the map sending x^n to the product

$$\underbrace{b_s b_t \dots}_{n \text{ factors}}$$

of alternating factors starting with b_s.

Proposition 2.6. *As a unital $\mathcal{A}$-algebra, $TL(X)$ is given by generators $\{b_s : s \in S\}$ and relations*

$$b_s^2 = \delta b_s, \tag{2}$$

$$b_s b_t = b_t b_s \quad \text{if } m(s,t) = 2, \tag{3}$$

$$(xP_{m-1})_b^{s,t}(x) = 0 \quad \text{if } 2 < m = m(s,t) < \infty, \tag{4}$$

where $\delta := (v + v^{-1})$.

Proof. This follows from [**12**, Corollary 6.5] and its proof, which shows that if $2 < m(s,t) < \infty$,

$$(xP_{m-1})_b^{s,t}(x)$$

is the image in $TL(X)$ of $C'_{w_{st}}$. (A similar result appears in [**8**, Proposition 9.5].) $\square$

Example 2.7. Relation (4) reads

$$\begin{aligned} b_s b_t b_s - b_s &= 0 \quad \text{if } m = 3, \\ b_s b_t b_s b_t - 2 b_s b_t &= 0 \quad \text{if } m = 4, \\ b_s b_t b_s b_t b_s - 3 b_s b_t b_s + b_s &= 0 \quad \text{if } m = 5, \text{ and} \\ b_s b_t b_s b_t b_s b_t - 4 b_s b_t b_s b_t + 3 b_s b_t &= 0 \quad \text{if } m = 6. \end{aligned}$$

Remark 2.8. Since the relations (3) all occur in $\mathrm{Co}(X, S)$, it makes sense, given an element $\mathbf{s} \in \mathrm{Co}(X, S)$ represented by a monomial $s_1 s_2 \cdots s_r$, to define an element $b(\mathbf{s}) \in TL(X)$ by

$$b(\mathbf{s}) := b_{s_1} b_{s_2} \cdots b_{s_r}.$$

The following lemma is the generalization of [**10**, Theorem 3.2.3] alluded to in [**10**, §4.1].

Lemma 2.9. *Let (W, S) be a star reducible Coxeter group, let $s_1 s_2 \cdots s_r$ be an arbitrary monomial in S^* representing the trace $\mathbf{s} \in \mathrm{Co}(X, S)$, and let $b(\mathbf{s})$ be the element of $TL(X)$ given in Remark 2.8. Express b as a linear combination of the monomial basis, namely*

$$b(\mathbf{s}) = \sum_{w \in W_c} \lambda_w b_w.$$

Then λ_w is an integer multiple of $\delta^{h(\mathbf{s})}$, where h is as in Theorem 2.1 and $\delta = (v + v^{-1})$.

Proof. We claim that $TL(X)$ has the structure of a graded $\mathbb{Z}$-module

$$\bigoplus_{k \geq 0} M_k,$$

where M_k is the free $\mathbb{Z}$-module on the set

$$\{\delta^p b(\mathbf{t}) \text{ such that } p \geq 0,\ \mathbf{t} \in \mathrm{Co}(X, S) \text{ and } p + h(\mathbf{t}) = k\}.$$

The only nontrivial thing to check is that the grading is respected by the relations of Proposition 2.6. Relation (3) clearly respects the grading, because it is a relation in $\mathrm{Co}(X, S)$. Relation (2) respects the grading by Theorem 2.1 (iii).

Note that relation (4) is a linear combination of monomials, each of which can be transformed into any of the others by iterated substitutions of the form $b_s b_t b_s \leftrightarrow b_s$ (see Example 2.7 for clarification). Although these substitutions are not generally valid relations in $TL(X)$, it now follows from Theorem 2.1 (iv) that relation (4) respects the grading given.

Now consider the monomial $b(\mathbf{s})$. By applying relations (2), (3) and (4) repeatedly to express b in terms of shorter monomials, we can write b as a linear combination

$$b(\mathbf{s}) = \sum_{w \in W_c} \lambda_w b_w,$$

where $\lambda_w = n_w \delta^{d_w}$ for some integer n_w and nonnegative integer d_w. By Theorem 2.1 (ii), all the monomials b_w in the sum are of the form $b(\mathbf{u})$, where $h(\mathbf{u}) = 0$. Since each side of the equation lies in $M_{h(\mathbf{s})}$, it follows that $d_w = h(\mathbf{s})$, as required. □

Lemma 2.10. *If W is a star reducible Coxeter group, then the b-basis and the c-basis of $TL(X)$ have the same $\mathbb{Z}$-span. In particular, the b-basis is an $\mathcal{A}^-$-basis for $\mathcal{L}$.*

Proof. Let $\mathbf{s} = s_1 s_2 \cdots s_r$ be a reduced expression for $w \in W_c$, and write

$$\begin{aligned} \widetilde{t}_w &= \widetilde{t}_{s_1} \widetilde{t}_{s_2} \cdots \widetilde{t}_{s_r} \\ &= (b_{s_1} - v^{-1})(b_{s_2} - v^{-1}) \cdots (b_{s_r} - v^{-1}). \end{aligned}$$

Expanding the parentheses, we express $\widetilde{t}_w$ as a linear combination of elements $(-v)^{-k} b(\mathbf{u})$, where $\mathbf{u}$ is obtained from $\mathbf{s}$ by deletion of k generators. By Theorem 2.1 (ii), $\mathbf{s}$ is acyclic, so by Theorem 2.1 (v), we must have $h(\mathbf{u}) \leq k$. By Lemma 2.9, if we express $(-v)^{-k} b(\mathbf{u})$ in terms of the monomial basis, namely

$$(-v)^{-k} b(\mathbf{u}) = \sum_{w \in W_c} (-v)^{-k} \lambda_w b_w,$$

we see that $(-v)^{-k}\lambda_w \in \mathcal{A}^-$.

It follows from this that $\widetilde{t}_w$ is an $\mathcal{A}^-$-linear combination of monomial basis elements. Since any monomial in the b_s is a linear combination of basis monomials of shorter length, the above argument shows that the coefficient of b_w in $\widetilde{t}_w$ is 1. This means that the change of basis matrix from the $\widetilde{t}$-basis to the b-basis is unitriangular with entries in $\mathcal{A}^-$ with respect to a suitable total ordering, and hence the inverse of this matrix has the same properties, in other words, the monomial basis elements lie in $\mathcal{L}$.

By Proposition 2.4, $\overline{b_w} = b_w$ for any $w \in W_c$. By Theorem 1.4 (iv), b_w is a $\mathbb{Z}$-linear combination of c-basis elements. By the above paragraph, we have

$$b_w = \widetilde{t}_w + \sum_{\substack{x \in W_c \\ x < w}} \nu_x \widetilde{t}_x$$

for certain $\nu_x \in \mathcal{A}^-$. Applying π to both sides and appealing to Theorem 1.4 (ii) and (iv), we have

$$b_w = c_w + \sum_{\substack{x \in W_c \\ x < w}} \xi_x c_x$$

for certain integers ξ_x. This shows that the change of basis matrix between the b-basis and the c-basis is unitriangular with entries in $\mathbb{Z}$ with respect to a suitable total ordering, from which it follows that the b-basis and the c-basis have the same $\mathbb{Z}$-span. This implies that they also have the same $\mathcal{A}^-$-span, namely $\mathcal{L}$. □

3. Monomials and weakly complex elements

In §3, we develop the properties of the lattice $\mathcal{L}$ by using the monomial basis which, as we know from Lemma 2.10, is an $\mathcal{A}^-$-basis for $\mathcal{L}$.

Lemma 3.1. *Let W be a star reducible Coxeter group. Then, for $s \in S$, the set*

$$\{x \in TL(X) : b_s x = (v + v^{-1})x\}$$

is the free $\mathcal{A}$-submodule of $TL(X)$ with basis $B_s := \{b_y : y \in W_c,\ sy < y\}$.

Proof. If $y \in W_c$ is such that $sy < y$, it is clear that $b_s b_y = \delta b_s$ by relation (2), and it follows that the set B_s is contained in the required subset of $TL(X)$.

To finish the proof, it is enough to show that if $b(\mathbf{u}) \in TL(X)$, then $b_s b(\mathbf{u})$ is a linear combination of elements b_y with $y \in W_c$ and $sy < y$.

Let us say that a monomial $\mathbf{s} = s_1 s_2 \cdots s_r \in S^*$ is "s-minimal" if the following conditions are satisfied:

1. $s_i = s$ for some $1 \leq i \leq r$;
2. $s_h \neq s$ for any $1 \leq h < i$;
3. s_h and s_i commute for any $1 \leq h < i$.

Condition 3 above means that it also makes sense to speak of an element $\mathbf{s} \in \mathrm{Co}(X, S)$ being s-minimal.

We see that applying one of the relations (2), (3) or (4) to $b(\mathbf{s})$ results in a linear combination of monomials $b(\mathbf{t})$ where $\mathbf{t}$ is also s-minimal. Repeating this argument shows that if $b(\mathbf{s})$ is s-minimal, then it is a linear combination of s-minimal basis elements. However, the s-minimal basis elements are precisely those basis elements b_y where y has a reduced expression beginning with s, which implies by relation (2) that $b_s b_y = \delta b_y$. Since any monomial of the form $b_s b(\mathbf{u})$ is s-minimal, the proof is complete. $\square$

Remark 3.2. It is tempting to think from Lemma 3.1 that if b_y is a monomial basis element such that $b_s b_y$ is δ times another basis element, then $sy < y$, but this is not true. If W is the (star reducible) Coxeter group of type B_3, and $S = \{s_1, s_2, s_3\}$ is indexed so that $m(s_1, s_2) = 4$ and $m(s_2, s_3) = 3$, then setting $y = s_1 s_2 s_1 s_3 \in W_c$ we have

$$b_{s_3} b_y = \delta b_z,$$

where $z = s_1 s_3 \in W_c$, even though $s_3 y > y$. The c-basis does not have this disadvantage, as will be clear from Theorem 4.6 (ii) below.

We recall the following definition from [**12**, §4].

Definition 3.3. Let $W' \subset W_c$. We define $\mathcal{L}^{W'}$ to be the free $\mathcal{A}^-$-module with basis

$$\{\widetilde{t}_w : w \in W'\} \cup \{v^{-1}\widetilde{t}_w : w \in W_c \backslash W'\}.$$

If $s, t \in S$ are noncommuting generators, $W_1 = \{w \in W_c : sw < w\}$ and $W_2 = \{w \in W_c : w = stu \text{ reduced}\}$, we write $\mathcal{L}_L^s$ and $\mathcal{L}_L^{st}$ for $\mathcal{L}^{W_1}$ and $\mathcal{L}^{W_2}$, respectively.

One can also define right handed versions, $\mathcal{L}_R^s$ and $\mathcal{L}_R^{ts}$, of the above concepts, and of Lemma 3.1.

Lemma 3.4. *Let W be a star reducible Coxeter group. Then the set*

$$\{b_y : y \in W_c,\ sy < y\} \cup \{v^{-1}b_z : z \in W_c,\ sz > z\}$$

is an $\mathcal{A}^-$-basis for $\mathcal{L}_L^s$.

Proof. Since the monomial basis is an $\mathcal{A}^-$-basis for $\mathcal{L}$ and there is a natural bijection between the set in the statement and the defining $\mathcal{A}^-$-basis for $\mathcal{L}_L^s$, the claim will follow if we can show that whenever we have $y \in W_c$ with $sy < y$, then

$$\pi(\widetilde{t}_y) = \pi(b_y) + \sum_{\substack{w<y \\ sw<w}} \xi_w \pi(b_w),$$

where $w \in W_c$ in the sum and $\xi_w \in \mathbb{Z}$. Apart from the assertion that $sw < w$, this follows from the observations relating the b-basis to the $\widetilde{t}$-basis made in the proof of Lemma 2.10.

Since $y = sy'$ is reduced, we have

$$\widetilde{t}_y = \widetilde{t}_s \widetilde{t}_{y'} = (b_s - v^{-1})\widetilde{t}_{y'},$$

and clearly $v^{-1}\widetilde{t}_{y'} \in v^{-1}\mathcal{L}$. Since $\widetilde{t}_y \in \mathcal{L}$, it follows that $b_s\widetilde{t}_{y'} \in \mathcal{L}$. However, by Lemma 3.1, we have

$$b_s \widetilde{t}_{y'} = \sum_{\substack{w \leq y \\ sw<w}} \lambda_w b_w,$$

where the sum is over $w \in W_c$ and we have $\lambda_w \in \mathcal{A}^-$ by Lemma 2.10. Since $\pi(\widetilde{t}_y) = \pi(b_s\widetilde{t}_{y'})$, the assertion follows. □

Lemma 3.5. *Let W be a star reducible Coxeter group, and let $s, t \in S$ be noncommuting generators. Then $b_s \mathcal{L}_L^t \subseteq \mathcal{L}_L^s$ and $\widetilde{t}_s \mathcal{L}_L^t \subseteq \mathcal{L}_L^s$.*

Proof. The second assertion is immediate from the first and the identity $b_s = (v^{-1}\widetilde{t}_1 + \widetilde{t}_s)$, so we concentrate on the first assertion.

Suppose that $y \in W_c$ is such that $ty < y$, and write $b_y = b(\mathbf{u})$ in the usual way, where $\mathbf{u} \in \mathrm{Co}(X, S)$. By Theorem 2.1 (ii), $h(\mathbf{u}) = 0$, and by Theorem 2.1 (i), $h(s\mathbf{u}) = 0$ too. Lemma 2.9 now shows that $b_s b_y$ is a $\mathbb{Z}$-linear combination of basis elements b_w, and then lemmas 2.10 and 3.1 show that $b_s b_y \in \mathcal{L}_L^s$.

Suppose now that $y \in W_c$ is such that $ty > y$, and write $b_y = b(\mathbf{u})$ as before. In this case, $h(\mathbf{u}) = 0$, and Theorem 2.1 (v) shows that $h(s\mathbf{u}) \leq 1$. Lemma 2.9 then shows that $b_s b_y$ is a $v\mathcal{A}^-$-linear combination of basis elements b_w. Lemmas 2.10 and 3.1 show that $b_s b_y \in v\mathcal{L}_L^s$.

An application of Lemma 3.4, combining the above two observations, completes the proof. □

To prove the main result of §3, we need to recall some of the combinatorial properties of weakly complex elements from [**12**]. The next result shows that weakly complex elements respect the left and right weak Bruhat orders.

Lemma 3.6. *Let W be any Coxeter group and let $w \in W_c$ be such that $sw \notin W_c$ for some $s \in S$. If $u \in S$ and $y \in W$ are such that we have either $w = uy$ or $w = yu$ reduced, then either $sy \in W_c$ or sy is weakly complex.*

Proof. See [**12**, Lemma 4.5 (iii)]. □

Lemma 3.7. *Let W be a star reducible Coxeter group, let $w \in W_c$ and $x = sw > w$, where $s \in S$. Then one of the following situations must occur:*

(i) *x is a product of commuting generators;*

(ii) *$x \in W_c$ and there exists $I = \{s, t\} \subseteq S$ with $st \neq ts$ such that when $x = x_I x^I$, we have $\ell(x_I) > 1$;*

(iii) *x is weakly complex and has a reduced expression begining with w_{st} for some $t \in S$ with $st \neq ts$;*

(iv) *there exists $I = \{u, u'\} \subset S$ with $s \notin I$, $uu' \neq u'u$, $su = us$ and $su' = u's$ such that when we write $w = w_I w^I$, we have $\ell(w_I) > 1$;*

(v) *there exists $I = \{u, u'\} \subset S$ with $uu' \neq u'u$ such that when we write $w = ({}^I w)({}_I w)$, we have $\ell({}_I w) > 1$;*

(vi) *x is weakly complex and there exist $t, u \in S$ with $st \neq ts$, $ut \neq tu$ and $su = us$ such that w has a reduced expression of the form*

$$u(tsts \cdots)x',$$

where the alternating product of t and s contains $m(s,t) - 1$ terms, and we have $u(tuw) > tuw$;

(vii) *x is weakly complex and there exist $t, u \in S$ with $m(s,t) = 3$, $ut \neq tu$ and $su = us$ such that $w = sx$ has a reduced expression of the form $w = utsux'$.*

Proof. This is [**12**, Lemma 6.9]. □

Lemma 3.8. *Let W be a star reducible Coxeter group and let $x \in W$ be a fully commutative or weakly complex element. Then we have:*

(i) *$\widetilde{t}_x \in \mathcal{L}$;*

(ii) *if $s \in S$ is such that $sx < x$, then $\widetilde{t}_x \in \mathcal{L}_L^s$;*

(iii) *if $s \in S$ is such that $xs < x$, then $\widetilde{t}_x \in \mathcal{L}_R^s$.*

Proof. The proof is by induction on $\ell(x)$, and the base case, $\ell(x) = 0$, is easy. In the inductive step, we will freely use the facts that, by Lemma 3.6, the elements sx and xs occurring in assertions (ii) and (iii) satisfy the inductive hypotheses.

We first prove assertion (i).

If $\ell(x) > 0$, we may use a case analysis based on Lemma 3.7 to prove the first assertion. If we are in case (i) of Lemma 3.7, this follows from the observation that if $x = s_1 s_2 \cdots s_r$ is a product of commuting generators, then $\widetilde{t}_x \in \mathcal{L}_L^s$ for each $s \in \{s_1, s_2, \dots, s_r\}$.

In case (ii) of Lemma 3.7, we may assume that x has a reduced expression beginning with st, where s and t are noncommuting generators. Since $tsx < sx$, we have $\widetilde{t}_{sx} \in \mathcal{L}_L^t$ by induction, and then $\widetilde{t}_x \in \mathcal{L}$ by Lemma 3.5. The analysis of case (iii) uses a similar argument.

In case (iv), we may assume that both sx and x have reduced expressions beginning uu', following the notation of Lemma 3.7. By induction, $\widetilde{t}_{ux} \in \mathcal{L}_L^{u'}$, and hence $\widetilde{t}_x \in \mathcal{L}$ by Lemma 3.5. The analysis of case (v) uses a similar argument.

In case (vi), we have $x = uw_{st}x'$ reduced, so that x has a reduced expression beginning ut. By induction, $\widetilde{t}_{ux} \in \mathcal{L}_L^t$, and hence $\widetilde{t}_x \in \mathcal{L}$ by Lemma 3.5. The analysis of case (vii) is the same, thus completing the proof of assertion (i).

We will now prove assertion (ii); the proof of assertion (iii) is by an analogous argument.

We know that $\widetilde{t}_{sx} \in \mathcal{L}$ by induction, and we have just shown that $\widetilde{t}_x \in \mathcal{L}$. Now

$$\widetilde{t}_x = \widetilde{t}_s \widetilde{t}_{sx} = (b_s - v^{-1})\widetilde{t}_{sx},$$

and we have $v^{-1}\widetilde{t}_{sx} \in v^{-1}\mathcal{L}$ from the definitions, which shows that

$$b_s \widetilde{t}_{sx} \in \mathcal{L}.$$

By Lemma 3.1, we have

$$b_s \widetilde{t}_{sx} = \sum_{w \in W_c} \lambda_w b_w,$$

where $\lambda_w \neq 0$ implies $sw < w$, and the fact that $b_s \widetilde{t}_{sx} \in \mathcal{L}$ means that all λ_w lie in $\mathcal{A}^-$. Lemma 3.4 shows that $b_s \widetilde{t}_{sx}$, and therefore $\widetilde{t}_s \widetilde{t}_{sx}$, lies in $\mathcal{L}_L^s$, as required. $\square$

Proposition 3.9. *Let W be a star reducible Coxeter group, let $s, t \in S$ be noncommuting generators and let $w \in W_c$. Then we have:*

(i)
$$\widetilde{t}_s \widetilde{t}_w \in \begin{cases} v\mathcal{L}_L^s & \text{if } sw < w, \\ \mathcal{L}_L^s & \text{if } sw > w; \end{cases}$$

(ii) *$\widetilde{t}_s\mathcal{L} \cap \mathcal{L} \subseteq \mathcal{L}^s_L$;*

(iii) *$\widetilde{t}_s\mathcal{L}^t_L \subseteq \mathcal{L}^{st}_L$.*

(iv) *if $a \in S$ does not commute with t and $a \neq s$, then $\widetilde{t}_a\mathcal{L}^{st}_L \subseteq \mathcal{L}^a_L$.* □

Proof. This was proved in [**12**, Proposition 4.10] for any Coxeter group satisfying the property that $\widetilde{t}_x \in \mathcal{L}^u_L$ whenever $x = uw$ is a weakly complex element, $w \in W_c$ and $u \in S$. This hypothesis is satisfied by Lemma 3.8 (ii). □

4. Main results

In §4, we will show that any element of a star reducible Coxeter group (not just a fully commutative element) has a reduced expression with a particularly nice form. More precisely, we have the following

Theorem 4.1. *Let W be a star reducible Coxeter group, and let $w \in W$. Then one of the following possibilities occurs for some Coxeter generators s, t, u with $m(s,t) \neq 2$, $m(t,u) \neq 2$ and $m(s,u) = 2$:*

(i) *w is a product of commuting generators;*

(ii) *w has a reduced expression beginning with st;*

(iii) *w has a reduced expression ending in ts;*

(iv) *w has a reduced expression beginning with sut.*

Proof. Let $\mathbf{s}$ be any reduced expression for w, and let

$$\mathbf{s}_1\mathbf{s}_2\cdots\mathbf{s}_p$$

be its Cartier–Foata normal form. If $p = 1$, then case (i) applies, and we are done.

If not, let t be a generator occurring in the factor $\mathbf{s}_2$. By definition of the normal form, t fails to commute with some generator in $\mathbf{s}_1$. If t fails to commute with only one such generator, s, then $\mathbf{s}$ is commutation equivalent to a reduced expression beginning with st, and case (ii) applies.

If t fails to commute with precisely two generators, s and u, in $\mathbf{s}_1$, then $\mathbf{s}$ is commutation equivalent to a reduced expression beginning sut, and we necessarily have $su = us$ by definition of the normal form, so case (iv) applies.

Note that there cannot be four distinct generators u_1, u_2, u_3, u_4 in $\mathbf{s}_1$ not commuting with t, or $u_1 u_2 t u_3 u_4$ would be an element of W_c that is neither star reducible nor a product of commuting generators, a contradiction. We may therefore assume that each generator t_i in $\mathbf{s}_2$ fails to commute with precisely three (necessarily distinct and mutually commuting) generators, $\{u_{ij} : 1 \leq j \leq 3\}$, in $\mathbf{s}_1$.

Suppose that $\mathbf{s}_2$ contains k generators and the set

$$\{u_{ij} : 1 \leq i \leq k,\ 1 \leq j \leq 3\}$$

consists of $3k$ distinct elements of $\mathbf{s}_1$. This implies that, given such a u_{ij}, the only generator in $\mathbf{s}_2$ not commuting with u_{ij} is t_i. Consequently, if $\mathbf{s}_3$ is empty, then w has a reduced expression ending in $u_{11} t_1$, and case (iii) applies. We may therefore assume that $\mathbf{s}_3$ contains a generator, t'. We know t' fails to commute with some element of $\mathbf{s}_2$, and without loss of generality, we may assume that $m(t', t_1) \neq 2$. None of the elements $\{t', u_{11}, u_{12}, u_{13}\}$ commutes with t_1, and if they were all distinct then

$$u_{11} u_{12} t_1 u_{13} t'$$

would be an element of W_c that would be neither star reducible nor a product of commuting generators, a contradiction. Without loss of generality, we may assume that $t' = u_{11}$, meaning that w has a reduced expression beginning

$$u_{12} u_{13} u_{11} t_1 u_{11}.$$

If $m(t_1, u_{11}) = 3$, we may apply a braid relation to transform this expression to one beginning

$$u_{12} u_{13} t_1 u_{11},$$

and case (iv) applies. If, on the other hand, $m(t_1, u_{11}) > 3$, the element

$$y = u_{12}u_{11}t_1u_{11}u_{13}$$

satisfies $y \in W_c$, but y is neither star reducible nor a product of commuting generators, a contradiction.

We have now reduced to the case where the set

$$\{u_{ij} : 1 \leq i \leq k,\ 1 \leq j \leq 3\}$$

is redundantly described. Without loss of generality, we may assume that $u := u_{11} = u_{21}$. Now

$$y' = u_{12}u_{13}t_1ut_2u_{22}u_{23}$$

lies in W_c, even if the set $\{u_{12}, u_{13}, u_{22}, u_{23}\}$ is redundantly described, because any two repeated occurrences of a generator s in the given reduced expression are separated by at least two occurrences of generators not commuting with s (see [**11**, Remark 3.3.2]). However, y' is neither a product of commuting generators, nor star reducible, so this case cannot occur, completing the analysis. □

Remark 4.2. By symmetry of the definitions, one can state a version of Theorem 4.1 in which condition (iv) is replaced by the condition "w has a reduced expression ending in tsu".

The following result, which was proved by Losonczy [**18**, Proposition 2.6, Theorem 3.4] in type D_n, is new in type E_n, type F_n $(n > 4)$, type H_n $(n > 4)$ and the two exceptional cases $\widetilde{E}_6$ and $\widetilde{F}_5$ discussed later (see Theorem 6.3).

Theorem 4.3. *If W is a star reducible Coxeter group and $\mathcal{L}_{\mathcal{H}}$ is the free $\mathcal{A}^-$-submodule of $\mathcal{H}$ with basis $\{\widetilde{T}_w : w \in W\}$, then the homomorphism*

$$\theta : \mathcal{H} \longrightarrow TL(X)$$

restricts to an $\mathcal{A}^-$-linear map from $\mathcal{L}_{\mathcal{H}}$ to $\mathcal{L}$. In particular, for any $w \in W$, we have $\theta(\widetilde{T}_w) \in \mathcal{L}$, and $\pi(\theta(\widetilde{T}_w)) = \pi(\theta(C'_w))$.

Proof. We first prove that $\widetilde{t}_w \in \mathcal{L}$ using induction on $\ell(w)$ and the case analysis of Theorem 4.1.

If w is a product of commuting generators, then $w \in W_c$ and the assertion is immediate from the definitions. This deals with the cases $\ell(w) \leq 1$.

If w has a reduced expression beginning with st, as in Theorem 4.1 (ii), then $\widetilde{t}_{sw}, \widetilde{t}_{tsw} \in \mathcal{L}$ by induction, and thus

$$\widetilde{t}_t \widetilde{t}_{tsw} = \widetilde{t}_{sw} \in \widetilde{t}_t \mathcal{L} \cap \mathcal{L} \subseteq \mathcal{L}_L^t$$

by Proposition 3.9 (ii). We therefore have

$$\widetilde{t}_s \widetilde{t}_{sw} = \widetilde{t}_w \in \widetilde{t}_s \mathcal{L}_L^t \subset \mathcal{L}$$

by Proposition 3.9 (iii), as required.

If w has a reduced expression ending in ts, as in Theorem 4.1 (iii), a symmetrical argument gives the desired conclusion.

Finally, suppose that w has a reduced expression beginning with sut, as in Theorem 4.1 (iv). By induction, $\widetilde{t}_{uw}, \widetilde{t}_{suw}, \widetilde{t}_{tsuw} \in \mathcal{L}$. We also have

$$\widetilde{t}_t \widetilde{t}_{tsuw} = \widetilde{t}_{suw} \in \widetilde{t}_t \mathcal{L} \cap \mathcal{L} \subseteq \mathcal{L}_L^t$$

by Proposition 3.9 (ii), and

$$\widetilde{t}_s \widetilde{t}_{suw} = \widetilde{t}_{uw} \in \widetilde{t}_s \mathcal{L}_L^t \subset \mathcal{L}_L^{st}$$

by Proposition 3.9 (iii). Finally, we have

$$\widetilde{t}_u \widetilde{t}_{uw} = \widetilde{t}_w \in \mathcal{L}_L^u \subset \mathcal{L}$$

by Proposition 3.9 (iv), as required.

This completes the proof that $\widetilde{t}_w \in \mathcal{L}$, and it is then clear that $\theta(\widetilde{T}_w) \in \mathcal{L}$. Since C'_w and $\widetilde{T}_w$ agree modulo $v^{-1}\mathcal{L}_{\mathcal{H}}$ (as explained in, for example, [**14**, Proposition 1.2.2]), the final claim also follows. $\square$

Lemma 4.4. *Let W be an arbitrary Coxeter group such that $I = \{s, t\} \subseteq S$ is a pair of noncommuting generators, and suppose that $w \in W$ satisfies $tw < w$ and $sw > w$.*

(i) *If $w \in W_c$, $tw < w$ and $sw \notin W_c$, then $sw = w_{st}w'$ is reduced.*

(ii) *Taking star operations with respect to I, we have*

$$C'_s C'_w = C'_{^*w} + C'_{_*w} \mod J(X),$$

where C'_z is defined to be zero if z is an undefined symbol.

Note. There is also a right-handed version of this result.

Proof. Part (i) follows from [**19**, Proposition 2.3] (see also [**12**, Lemma 4.5 (i)]), and part (ii) follows from [**12**, Lemma 6.2]. $\square$

Lemma 4.5. *Let W be a star reducible Coxeter group and let $x \in W$ be weakly complex. Then $\theta(C'_x) = 0$.*

Proof. We write $x = sw$ with $s \in S$ and $w \in W_c$. The proof is by induction on $\ell(x)$, using Lemma 3.6 and the case analysis of Lemma 3.7.

Since x is weakly complex, we are in one of cases (iii)–(vii) of Lemma 3.7. Let us first suppose we are in case (iii), meaning that $x = w_{st}x'$ is reduced. Since x has a reduced expression beginning with st and $\widetilde{t}_x, \widetilde{t}_{sx} \in \mathcal{L}$ by Theorem 4.3, Proposition 3.9 (ii) shows that $\widetilde{t}_x \in \mathcal{L}^s_L$. Similarly, x has a reduced expression beginning with ts, and $\widetilde{t}_x \in \mathcal{L}^t_L$. Since s and t do not commute, we have $\mathcal{L}^s_L \cap \mathcal{L}^t_L \subseteq v^{-1}\mathcal{L}$, which shows that $\pi(\widetilde{t}_x) = 0$. By Theorem 4.3, we also have $\pi(\theta(C'_x)) = 0$. Since $\overline{\theta(C'_x)} = \theta(C'_x)$ and $\theta(C'_x) \in \mathcal{L}$, [**15**, Lemma 2.2.2] shows that $\theta(C'_x) = 0$, as required.

Suppose that we are in case (iv) of Lemma 3.7. We may assume without loss of generality that w and x each have a reduced expression beginning uu', where $I' = \{u, u'\}$ is a pair of noncommuting generators and u, u' satisfy the conditions of Lemma 3.7 (iv). We cannot have $x = w_{uu'}x'$ reduced, or $w = sx = w_{uu'}(sx')$ would not be fully commutative, which is a contradiction. Taking star operations with respect to I', we may therefore assume that ${}^*ux$ is defined and equal to x, and furthermore (by Lemma 4.4 (i)), that ux is weakly complex. By Lemma 4.4 (ii), we then have

$$C'_u C'_{ux} = C'_x + C'_{{}_*ux} \mod J(X).$$

Since $C'_{ux} \in J(X)$ by induction, we need to show that $C'_{{}_*ux} \in J(X)$. We may assume that ${}_*ux$ is defined, or this is obvious. By Lemma 3.6, either ${}_*ux$ is weakly complex or fully commutative, and in the former case we are done by the inductive hypothesis. However, if ${}_*ux \in W_c$, then the fact that $ux \notin W_c$ implies by Lemma 4.4 (i) that $u.ux = x < ux$, a contradiction. This completes the analysis of case (iv), and case (v) follows by a similar argument. The only difference in the argument needed to treat case (v) is that we may have $x = x'w_{uu'}$ reduced, in which case we are done by an argument like that used to treat case (iii).

Suppose we are in case (vi) of Lemma 3.7, and keep the same notation. In this case, we have $x = uw_{st}x'$ reduced, and furthermore, w_{st} has a reduced expression beginning with t, which does not commute with u. As in case (iii), we may assume that we do not have $x = w_{tu}x'$ reduced. Taking star operations with respect to $I'' = \{u, t\}$, we may assume as in the analysis of case (iv) that ${}^*ux$ is defined and equal to x, and that ux is weakly complex. By Lemma 4.4 (ii), we now have

$$C'_u C'_{ux} = C'_x + C'_{{}_*ux} \mod J(X).$$

As in the analysis of case (iv), the only nonobvious case left to consider is when ${}_*ux$ is defined and fully commutative. In this case, ${}_*ux$ is reduced of the form

$$(stst\cdots)x',$$

where there are $m(s,t) - 1$ occurrences of s or t. However, this cannot happen: $t \in \mathcal{L}(ux)$ implies that $u \in \mathcal{L}({}_*ux)$, and a fully commutative element cannot have a reduced expression beginning with st and another beginning with u if $s \neq u$ and t and u do not commute.

The analysis for case (vii) is exactly the same as that for case (vi), and this completes the proof. □

Theorem 4.6. *Let W be a star reducible Coxeter group.*

(i) *If $w \in W$ is weakly complex, then $\widetilde{t}_w \in v^{-1}\mathcal{L}$; in other words, W has "Property W", in the sense of* [**12**].

(ii) *If $w \in W_c$, then we have*

$$c_s c_w = \begin{cases} (v+v^{-1})c_w & \text{if } \ell(sw) < \ell(w), \\ c_{sw} + \sum_{sy<y} \mu(y,w) c_y & \text{if } \ell(sw) > \ell(w), \end{cases}$$

where c_z is defined to be zero whenever $z \notin W_c$, and where $\mu(y,w)$ is the integer defined by Kazhdan and Lusztig in [**17**].

(iii) *If $I = \{s,t\}$ is a pair of noncommuting generators, and we have $w \in W_c$ with $tw < w$, then we have*

$$c_s c_w = c_{{}^*w} + c_{{}_*w},$$

where c_z is defined to be zero whenever z is an undefined symbol.

(iv) *If $w \in W_c$, then $c_w = \theta(C'_w)$.*

(v) *The structure constants arising from the c-basis of $TL(X)$ lie in $\mathbb{Z}^{\geq 0}[\delta]$.* □

Proof. For part (i), let $w \in W$ be a weakly complex element. We know from Theorem 4.3 that $\pi(\theta(\widetilde{T}_w)) = \pi(\theta(C'_w))$, and we know from Lemma 4.5 that $\pi(\theta(C'_w)) = 0$. Part (i) is immediate from these observations.

Part (ii) is essentially [**12**, Theorem 5.13], the only difference being that (i) allows us to remove the extra hypothesis that W should have Property W. Similarly, parts (iii) and (iv) now follow from [**12**, Proposition 6.3], and part (v) now follows from [**12**, Theorem 6.13]. □

Remark 4.7. Note that part (iii) of the theorem allows the c-basis to be constructed inductively. Part (v) proves [**14**, Conjecture 1.2.4] for star reducible Coxeter groups. This is a new result for type F_n $(n > 4)$ and type $\widetilde{F}_5$ (see Lemma 5.5), and it provides a new elementary proof of positivity in type $\widetilde{C}_{n-1}$ (for n even).

5. Some examples of star reducible Coxeter groups

In §5, we present some specific examples of star reducible Coxeter groups, and we present various methods to construct new examples out of known ones. It will turn out in §6 that these methods suffice to construct all examples, assuming as always that the Coxeter generating set S is finite.

In order to show that certain Coxeter groups are star reducible, we need to associate a sequence of graphs to each Cartier–Foata normal form. This idea has also been used by Fan in [**5**, Lemma 4.3.2], and by Fan and the author in [**6**, §2.4].

Definition 5.1. Let $\mathbf{s}$ be an element of the commutation monoid $\mathrm{Co}(X, S)$ with Cartier–Foata normal form $\mathbf{s} = \mathbf{s}_1\mathbf{s}_2 \cdots \mathbf{s}_p$. For all $1 \leq i < p$, we define the graph $X_i(\mathbf{s})$ to be the induced labelled subgraph of X corresponding to the set of all generators appearing in the factors $\mathbf{s}_i$ and $\mathbf{s}_{i+1}$. If $w \in W_c$, then we define $X_i(w)$ to be the graph $X_i(\mathbf{s})$, where $\mathbf{s}$ is the (unique) element of $\mathrm{Co}(X, S)$ corresponding to w.

Remark 5.2. If $\mathbf{s}$ is a reduced expression for some Coxeter group element, the generators appearing in the subword $\mathbf{s}_i\mathbf{s}_{i+1}$ of $\mathbf{s}$ are distinct, by definition of the normal form.

For the next lemma, we assume that the Coxeter group (W, S) is of type $\widetilde{C}_{2l+1}$ ($l \geq 1$), meaning that $S = \{s_1, s_2, \ldots, s_{2l+2}\}$ and we have the relations

(a) $m(s_i, s_j) = 2$ if $|i - j| > 1$,
(b) $m(s_1, s_2) = m(s_{2l+1}, s_{2l+2}) = 4$,
(c) $m(s_i, s_{i+1}) = 3$ if $1 < i < 2l + 1$.

Lemma 5.3. *Let W be the Coxeter group of type $\widetilde{C}_{2l+1}$, with the above notation. Suppose that $\mathbf{s} \in \mathrm{Co}(X, S)$ corresponds to a reduced expression for $w \in W_c$, and let $\mathbf{s}_1\mathbf{s}_2 \cdots \mathbf{s}_p$ be the Cartier–Foata normal form of $\mathbf{s}$. Suppose also that $w \in W_c$ is not left star reducible. Then, for $1 \leq i < p$ and $1 \leq j \leq 2l + 2$, the following hold:*

(i) *if s_1 occurs in $\mathbf{s}_{i+1}$, then s_2 occurs in $\mathbf{s}_i$;*

(ii) *if* s_{2l+2} *occurs in* $\mathbf{s}_{i+1}$*, then* s_{2l+1} *occurs in* $\mathbf{s}_i$*;*

(iii) *if* $j \notin \{1, 2l+2\}$ *and* s_j *occurs in* $\mathbf{s}_{i+1}$*, then both* s_{j-1} *and* s_{j+1} *occur in* $\mathbf{s}_i$*.*

Proof. The assertions of (i) and (ii) are immediate from properties of the normal form, because s_2 (respectively, s_{2l+1}) is the only generator not commuting with s_1 (respectively, s_{2l+2}). We will now prove (iii) by induction on i. Suppose first that $i = 1$.

Suppose that $j \notin \{1, 2l+2\}$ and that s_j occurs in $\mathbf{s}_2$. By definition of the normal form, there must be a generator $s \in \mathbf{s}_1$ not commuting with s_j. Now s cannot be the only such generator, or w would be left star reducible to $sw < w$. Since the only generators not commuting with s_j are s_{j-1} and s_{j+1}, these must both occur in $\mathbf{s}_1$.

Suppose now that the statement is known to be true for $i \leq N$, and let $i = N+1 \geq 2$. Suppose also that $j \notin \{1, 2l+2\}$ and s_j occurs in $\mathbf{s}_{N+1}$. As in the base case, there must be at least one generator s occurring in $\mathbf{s}_N$ that does not commute with s_j.

Let us first consider the case where $j \notin \{2, 2l+1\}$, and write $s = s_k$ for some $1 \leq k \leq 2l+2$. The restrictions on j mean that $2 < k < 2l+1$. By the inductive hypothesis, this means that s_{k-1} and s_{k+1} both occur in $\mathbf{s}_{N-1}$, and that $m(s_{k-1}, s_k) = m(s_k, s_{k+1}) = 3$. Now either $j = k-1$ or $j = k+1$; we consider the first possibility, the other being similar. (Since $j \geq 3$, this means $k \geq 4$.) If s_{k-2} occurs in $\mathbf{s}_N$, then statement (i) follows as both generators not commuting with s_j lie in $\mathbf{s}_N$. If, on the other hand, s_{k-2} does not occur in $\mathbf{s}_N$, the fact that s_{k-1} occurs both in $\mathbf{s}_{N-1}$ and in $\mathbf{s}_{N+1}$ means that the word $\mathbf{s}$ can be parsed in the form $\mathbf{u}_1 s_{k-1} \mathbf{u}_2 s_{k-1} \mathbf{u}_3$, where all the generators in $\mathbf{u}_2$ commute with s_{k-1} except for one occurrence of s_k. This means that $\mathbf{s}$ is represented by a word in S^* containing a subword $s_{k-1} s_k s_{k-1}$, which contradicts the assumption $w \in W_c$.

Now suppose that $j = 2$ (the case $j = 2l+1$ follows by a symmetrical argument). If both s_1 and s_3 occur in $\mathbf{s}_N$, then we are done. If s_3 occurs in $\mathbf{s}_N$ but s_1 does not, then the argument of the previous paragraph applies. Suppose then that s_1 occurs in $\mathbf{s}_N$ but s_3 does not. By statement (i), s_2 occurs in $\mathbf{s}_{N-1}$, and we cannot have $N = 2$, or w would be left star reducible to $s_2 w < w$. Applying the inductive hypothesis to (i), we see that s_1 and s_3 both occur in $\mathbf{s}_{N-2}$. Putting all this together, we find that w has a reduced expression containing a subword of the form $s_3 s_1 s_2 s_1 s_2$, which is incompatible with $w \in W_c$. This completes the inductive step. □

Proposition 5.4. *A Coxeter group of type* $\widetilde{C}_{2l+1}$ *is star reducible.*

Proof. Keeping the previous notation, we suppose that $w \in W_c$ is not left star reducible and prove that either w is a product of commuting generators, or w is right star reducible.

If $\mathbf{s}_2$ is empty, then w is a product of commuting generators, and we are done. Otherwise, the graph $X_{p-1}(w)$ has the property that not all of its connected components have size 1. Let Γ be one of the components with $|\Gamma| > 1$.

Suppose that $\Gamma = \widetilde{C}_{2l+1}$, which has an even number of vertices. Either this forces s_1 to occur in $\mathbf{s}_{p-1}$ and s_2 to occur in $\mathbf{s}_p$, or it forces s_{2l+2} to occur in $\mathbf{s}_{p-1}$ and s_{2l+1} to occur in $\mathbf{s}_p$. In the first case, w is right star reducible with respect to $\{s_1, s_2\}$, and in the second, w is right star reducible with respect to $\{s_{2l+1}, s_{2l+2}\}$.

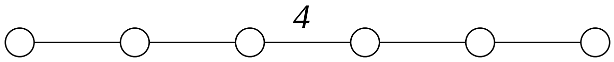
Suppose that Γ is a Coxeter graph of type B_n. Conditions (i)–(iii) of Lemma 5.3 show that there are four possibilities:

(a) s_1 occurs in $\mathbf{s}_{p-1}$ and corresponds to a vertex of Γ, and n is odd;
(b) s_{2l+2} occurs in $\mathbf{s}_{p-1}$ and corresponds to a vertex of Γ, and n is odd;
(c) s_1 occurs in $\mathbf{s}_p$ and corresponds to a vertex of Γ, and n is even;
(d) s_{2l+2} occurs in $\mathbf{s}_p$ and corresponds to a vertex of Γ, and n is even.

Let $k > 1$ be the number of vertices in Γ. In case (a), w is right star reducible with respect to $\{s_1, s_2\}$, and in case (b), with respect to $\{s_{2l+1}, s_{2l+2}\}$. In case (c), w is right star reducible with respect to $\{s_{k-1}, s_k\}$, and in case (d), with respect to $\{s_{2l+3-k}, s_{2l+4-k}\}$.

FIGURE 1. The Coxeter graph X of Lemma 5.5

$\tilde{F}_5$



The only other possibility is that Γ is a Coxeter graph of type A_k. In this case, condition (iii) of Lemma 5.3 forces $k > 1$ to be odd. If $\{s_a, s_{a+1}, \ldots, s_b\}$ are the generators involved in Γ, then s_a and s_b both lie in $\mathbf{s}_{p-1}$, and w is right star reducible with respect to $\{s_a, s_{a+1}\}$ and with respect to $\{s_{b-1}, s_b\}$. $\square$

Lemma 5.5. *Let W be the Coxeter group with Coxeter matrix*

$$(m_{i,j})_{1 \leq i,j \leq 6} = \begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 3 & 1 & 4 & 2 & 2 \\ 2 & 2 & 4 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 & 1 & 3 \\ 2 & 2 & 2 & 2 & 3 & 1 \end{pmatrix},$$

and denote $S = \{s_1, s_2, \ldots, s_6\}$ in the obvious way. Then W is star reducible.

Note. The graph X in this case is shown in Figure 1. Note that there is a symmetry of the graph X, namely that sending s_i to s_{7-i}, which induces a Coxeter group automorphism of $W(X)$.

Proof. Let $w \in W_c$ be such that w is not left star reducible or a product of commuting generators, and suppose (for a contradiction) that w is not right star reducible.

Let $\mathbf{s} \in \mathrm{Co}(X, S)$ correspond to w, and let

$$\mathbf{s} = \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_p$$

be the corresponding Cartier–Foata normal form. Since w is not a product of commuting generators, there exists a generator $s_k \in \mathbf{s}_2$. Since w is not left star reducible, there must be at least two generators in $\mathbf{s}_1$ that do not commute with s. Because X is a straight line, these two generators must be s_{k-1} and s_{k+1}, so that in particular we cannot have $k = 1$ or $k = 6$. Since $|X| = 6$ and the generators from $\mathbf{s}_1$ pairwise commute, we must therefore have $2 \leq |\mathbf{s}_1| \leq 3$.

Suppose first that $|\mathbf{s}_1| = 2$. By symmetry of X and the above remarks, it suffices to consider the cases $\mathbf{s}_1 = s_1 s_3$ and $\mathbf{s}_1 = s_2 s_4$.

If $\mathbf{s}_1 = s_1 s_3$ then $\mathbf{s}_2$ can only contain s_2, for if $\mathbf{s}_2$ contained s_4 (the only other generator not commuting with either s_1 or s_3) then w would be left star reducible to $s_3 w$. Now $s_1 s_3 s_2$ is right star reducible, so $\mathbf{s}_3$ must contain a generator, and this generator must not commute with s_2. We cannot have s_1 occurring in $\mathbf{s}_3$, or w would have a reduced expression containing $s_1 s_2 s_1$ consecutively. Similarly, we cannot have s_3 occurring in $\mathbf{s}_3$, producing a contradiction.

If $\mathbf{s}_1 = s_2 s_4$ then, arguing as in the above paragraph, we find that $\mathbf{s}_2 = s_3$, $\mathbf{s}_3 = s_4$, $\mathbf{s}_4 = s_5$ and $\mathbf{s}_5 = s_6$. At this point, we are stuck, and $s_2 s_4 s_3 s_4 s_5 s_6$ is right star reducible, which is a contradiction.

Suppose now that $|\mathbf{s}_1| = 3$. By symmetry of X, we may assume that $\mathbf{s}_1 = s_1 s_3 s_5$. Now $\mathbf{s}_2$ is nonempty, but it cannot contain s_6, or w would be left star reducible to $s_5 w$. If $\mathbf{s}_2$ contains only s_4, then $\mathbf{s}_3$ must be nonempty as $s_1 s_3 s_5 s_4$ is right star reducible. In turn, we must have $\mathbf{s}_3 = s_3$, $\mathbf{s}_4 = s_2$, $\mathbf{s}_5 = s_1$, and then there are no possible choices for $\mathbf{s}_6$, a contradiction. If $\mathbf{s}_2$ contains only s_2, then a similar argument shows that all choices for $\mathbf{s}_3$ lead to a contradiction. The only other possibility is for $\mathbf{s}_2 = s_2 s_4$, which forces $\mathbf{s}_3 = s_3$. However, $s_1 s_3 s_5 s_2 s_4 s_3$ is right star reducible, and we must then have $\mathbf{s}_4 = s_4$, $\mathbf{s}_5 = s_5$ and then there are no possible choices for $\mathbf{s}_6$, a contradiction. We have exhausted all the possibilities, so the assumption that w is not right star reducible is wrong, completing the proof. □

Lemma 5.6. *If W is a Coxeter group for which W_c is finite, then W is star reducible.*

Proof. As pointed out in [**12**, Remark 3.5], this result follows from the argument of [**5**, Lemma 4.3.1] together with [**19**, Proposition 2.3].

The way the argument works is as follows. Suppose that $w_1 \in W_c$ has the property that w_1 is neither left nor right star reducible. Let $\mathbf{s}$ be a reduced expression for w_1, and let

$$\mathbf{s} = \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_p$$

be the Cartier–Foata normal form of the corresponding element of $\mathrm{Co}(X, S)$. The results of Fan and Stembridge just mentioned show that

$$\mathbf{s}_p \mathbf{s}_{p-1} \cdots \mathbf{s}_2 \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_{p-1} \mathbf{s}_p$$

is also a reduced expression for an element $w_2 \in W_c$ that also has the property that it cannot be left or right star reduced. Proceeding in this way, we obtain an infinite sequence $\{w_i\}_{i \in \mathbb{N}}$ of distinct elements of W_c, which contradicts the hypothesis. $\square$

Lemma 5.7. *If W is a Coxeter group for which no two distinct elements of S commute, then W is star reducible.*

Proof. Let $w \in W_c$. The hypotheses show that w has a unique reduced expression,

$$w = s_1 s_2 \cdots s_k.$$

Since s_1 and s_2 do not commute by hypothesis, w is star reducible to $s_1 w < w$. Iterating this argument proves the assertion. $\square$

The following useful lemma is an easy consequence of the definitions.

Lemma 5.8. *If (W, S) is a star reducible Coxeter group, then so is any parabolic subgroup (W_I, I) of (W, S). In particular, any connected component of the Coxeter graph of a star reducible Coxeter group corresponds to another star reducible Coxeter group.* $\square$

Definition 5.9. Let (W, S) be a Coxeter group corresponding to Coxeter graph X and function $m : S \times S \longrightarrow \mathbb{N}$. We define the Coxeter group $(v(W), S)$ to be the group corresponding to the function $m' : S \times S \longrightarrow \mathbb{N}$, where

$$m'(i,j) = \begin{cases} m(i,j) & \text{if } m(i,j) < 3; \\ 3 & \text{otherwise.} \end{cases}$$

In other words, it is the group obtained by deleting all edge labels bigger than 3 (including edges with infinite label) in X.

Lemma 5.10. *If (W, S) is a star reducible Coxeter group, then so is $(\upsilon(W), S)$.*

Proof. Let $w \in \upsilon(W)$ be a fully commutative element, and let $\mathbf{s}$ be a reduced expression for w. Since all reduced expressions for w are commutation equivalent, and since two generators $s, s' \in S$ commute in $\upsilon(W)$ if and only if they commute in W, it follows that $\mathbf{s}$ is also a reduced expression for a fully commutative element $w^+ \in W$.

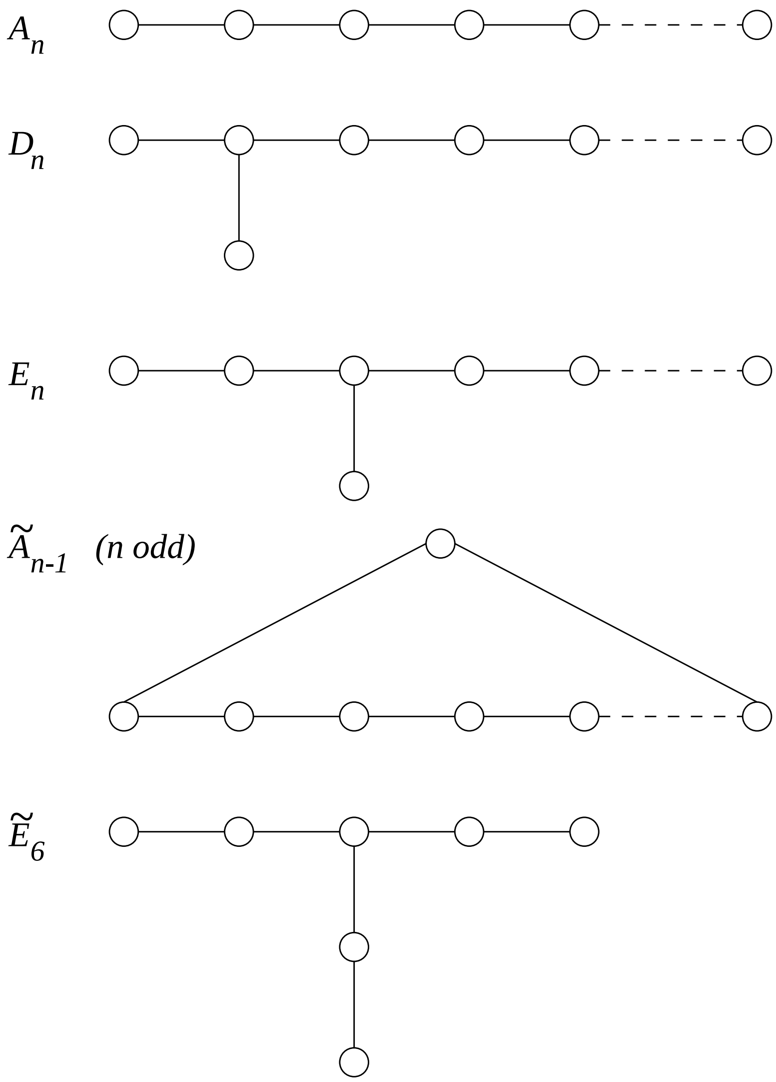
Since W is star reducible, either w^+ is a product of commuting generators in W (which means that w is a product of commuting generators in $\upsilon(W)$), or w^+ is left or right star reducible to some other element of W. We treat the case of left star reducibility, since the other case is similar. Suppose that w^+ is left star reducible with respect to $I = \{s, s'\} \subseteq S$. If $m, m' : S \times S \to \mathbb{N}$ are the functions arising from the Coxeter groups (W, S) and $(\upsilon(W), S)$ respectively, then we have $m(s, s') \geq m'(s, s') \geq 3$ by Definition 5.9 and the fact that s, s' do not commute. This means that we can identify the $\{s, s'\}$-string, S_w, in $\upsilon(W)$ containing w with a subset of the $\{s, s'\}$-string, S_{w^+}, in W containing w^+; here S_w will consist of the $(m'(s, s') - 1)$ shortest elements of S_{w^+}. Since star reducibility moves w^+ to a shorter element in S_{w^+}, there is a corresponding star reduction of w to a shorter element in S_w. By iterating this procedure, we see that w can be star reduced to a product of commuting generators, as required. □

The benefit of Lemma 5.10 is that the simply laced star reducible Coxeter groups have already been classified [**11**].

Theorem 5.11 [**11**]. *Let W be a simply laced Coxeter group with (finite) generating set S. Then W is star reducible if and only if each component of X is either a complete graph K_n or appears in the list depicted in Figure 2: type A_n $(n \geq 1)$, type D_n $(n \geq 4)$, type E_n $(n \geq 6)$, type $\widetilde{A}_{n-1}$ $(n \geq 3$ and n odd$)$ or type $\widetilde{E}_6$.*

Note. The corresponding result for arbitrary $|S|$ is not much more difficult, but we do not state it in order to avoid cardinality issues.

FIGURE 2. Connected incomplete graphs associated to simply laced star reducible Coxeter groups

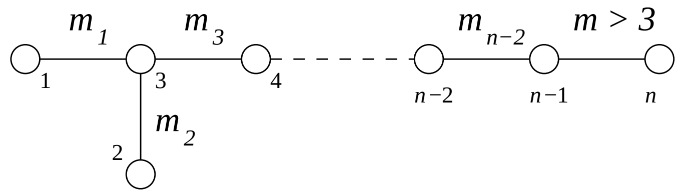


Proof. This is a restatement of [**11**, Theorem 1.5.2] using the definitions and remarks of [**11**, §1.2]. $\square$

6. Classification of star reducible Coxeter groups

We are now ready to classify the star reducible Coxeter groups (W, S) for finite S. During the argument, which is reminiscent of the classification of finite Coxeter groups [**16**, §2] and the classification of FC-finite Coxeter groups (see [**19**, §4], [**8**,

FIGURE 3. Coxeter graphs considered in the proof of Lemma 6.1



§7]), we will freely use the contrapositive statement to Lemma 5.8.

By Lemma 5.10 and Theorem 5.11, the remaining part of this task will be to determine how the edge labels in the graphs listed in Figure 2 may be increased so as to obtain another star reducible Coxeter group. We first deal with the case where the graph has a branch point, which means that it is of type D_n, E_n or $\widetilde{E}_6$.

Lemma 6.1. *Suppose that X is a connected Coxeter graph with a branch point, and that $W(X)$ is star reducible. Then X is simply laced.*

Proof. By the remarks preceding the statement (and Lemma 5.8), it is enough to show that X cannot arise from a graph of Coxeter type D_n, where the label of the edge furthest from the branch point is greater than 3, and where some of the other edges with labels $m \geq 3$ may also have been increased; see Figure 3. (If $n = 4$, the condition is that at least one of the edge labels must strictly exceed 3.)

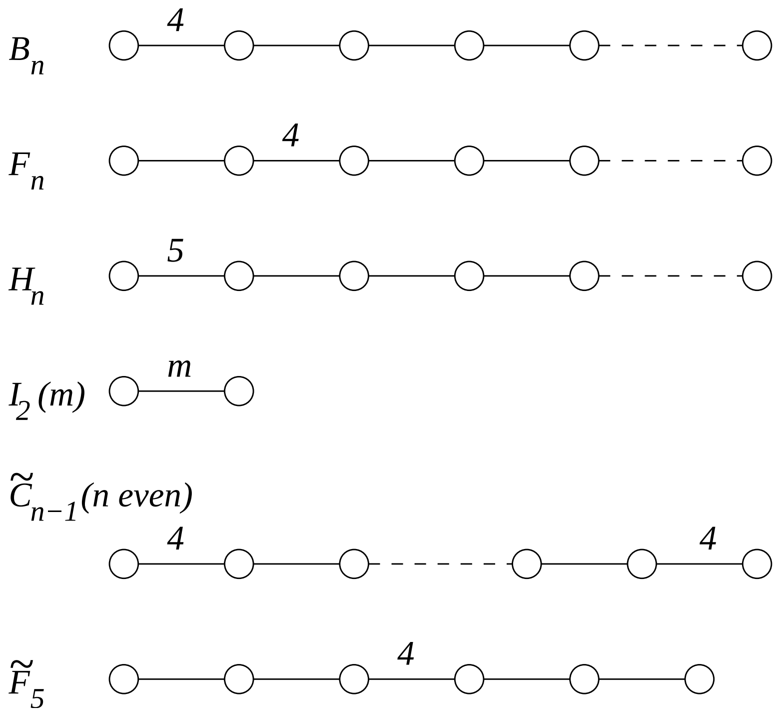
Labelling the vertices as in Figure 3 (where vertices 1 and 2 commute, 3 is the branch point, and $m(n-1, n) > 3$), we find that

$$(s_1 s_2) s_3 s_4 \cdots s_{n-2} s_{n-1} s_n s_{n-1} s_{n-2} \cdots s_4 s_3 (s_1 s_2)$$

is a fully commutative element that cannot be left or right star reduced, but that is not a product of commuting generators, which completes the proof. $\square$

Lemma 6.2. *Suppose that X is a Coxeter graph whose unlabelled graph is a k-cycle, where $k \geq 5$ is odd, and that $W(X)$ is star reducible. Then X is simply laced.*

FIGURE 4. Connected incomplete graphs associated to non simply laced star reducible Coxeter groups



Proof. Numbering the Coxeter generators $s_1, s_2, \ldots, s_k$ in an obvious cyclic fashion, let us assume that $m(s_k, s_1) > 3$. Since $k \geq 5$, we have $m(s_2, s_k) = 2$ and $m(s_1, s_{k-1}) = 2$. In this case, the element

$$(s_2 s_k) s_1 s_k s_{k-1} s_{k-2} \cdots s_3 s_2 s_1 s_k (s_{k-1} s_1)$$

is a fully commutative element that cannot be left or right star reduced, but that is not a product of commuting generators. □

Finally, we may classify all star reducible Coxeter groups with a finite generating set.

Theorem 6.3. *Let $W(X)$ be a Coxeter group with (finite) generating set S. Then $W(X)$ is star reducible if and only if each component of X is either a complete graph with all labels $m(i,j) \geq 3$, or appears in one of the lists depicted in Figure 2 or Figure 4: type A_n $(n \geq 1)$, type B_n $(n \geq 2)$, type D_n $(n \geq 4)$, type E_n $(n \geq 6)$,* *type F_n ($n \geq 4$), type H_n ($n \geq 2$), type $I_2(m)$ ($m \geq 3$), type $\widetilde{A}_{n-1}$ ($n \geq 3$ and n odd), type $\widetilde{C}_{n-1}$ ($n \geq 4$ and n even), type $\widetilde{E}_6$ or type $\widetilde{F}_5$.*

Note. Although there appear to be ten infinite families in the classification above, the family $I_2(m)$ consists entirely of complete graphs and may thus be incorporated into another family.

Proof. We first summarize why the examples listed are star reducible. The families A, B, D, E, F, H, I have the property that W_c is finite (see [**19**, §4], [**8**, §7]), so they are star reducible by Lemma 5.6. Types $\widetilde{A}_{n-1}$ and $\widetilde{E}_6$ are covered by Theorem 5.11, type $\widetilde{C}_{n-1}$ is covered by Proposition 5.4, and type $\widetilde{F}_5$ is covered by Lemma 5.5.

Let us now prove that the list given is complete, bearing in mind that Lemma 5.8 allows us to reduce consideration to connected components. If W is star reducible, Lemma 5.10 shows that $v(W)$ is as well. If the graph X is complete, then any increased labels are permissible by Lemma 5.7, so our list of complete graphs is correct.

There is no way to increase the labels of edges of the graphs of types D, E or $\widetilde{E}_6$ appearing in Figure 2 by Lemma 6.1, so our list of graphs with branch points is complete.

If the Coxeter graph X is a cycle and W is star reducible, it must be a cycle of odd length by Lemma 5.10 and Theorem 5.11. A cycle of length 3 is a complete graph, and then any labels are permissible. A cycle of length 5 or greater cannot have any labels increased by Lemma 6.2, so our list of cycle shaped graphs is complete.

We have reduced consideration to the case where X is a straight line. Let us label the Coxeter generators $s_1, s_2, \ldots, s_n$ in an obvious way. We shall assume that $n \geq 3$, or else X is complete, which we have dealt with above.

We first show that X has no edge labelled 6 or greater. To check this, it is enough by Lemma 5.8 to consider the case where $n = 3$ and $m(s_2, s_3) \geq 6$. In this case, the element

$$s_1 s_3 s_2 s_3 s_2 s_1 s_3$$

provides the required counterexample of a fully commutative element that is not a product of commuting generators, but also not left or right star reducible.

Suppose now that X has an edge labelled 5 (but no labels strictly greater than 5, by the above). We claim that this edge must be extremal. If not, we may reduce to the case where $n = 4$ and $m(s_2, s_3) = 5$. In this case,

$$s_1 s_3 s_2 s_3 s_2 s_4$$

provides the required counterexample.

Suppose that X has an extremal edge labelled 5. In this case, we claim that this edge is the only edge with a label greater than 3. If not, we may reduce (using Lemma 5.8 as always) to the case where $m(s_1, s_2) = 5$ and $m(s_{n-1}, s_n) > 3$. In this case, the element

$$s_1 s_3 s_2 s_1 s_2 s_3 s_4 \cdots s_{n-1} s_n s_{n-1} \cdots s_4 s_3 s_2 s_1 s_2 s_1 s_3$$

provides the required counterexample. We conclude that if X has an edge with label 5, then X is of type H_n, which is on the list.

Suppose now that X has at least two edges labelled 4, but no edge with label 5 or higher. If one of these edges is not extremal, then we may reduce to the case where $m(s_2, s_3) = 4$ and $m(s_{n-1}, s_n) = 4$, and

$$s_1 s_3 s_2 s_3 s_4 \cdots s_{n-1} s_n s_{n-1} \cdots s_4 s_3 s_2 s_1 s_3$$

provides the required counterexample. We deduce that there are precisely two edges labelled 4, and that they are both extremal.

We claim that the two edges labelled 4 in the above paragraph must have an odd number of other edges between them. If not, we may reduce to the case where n is odd and $m(s_1, s_2) = m(s_{n-1}, s_n) = 4$, and now

$$(s_1 s_3 s_5 \cdots s_n)(s_2 s_4 s_6 \cdots s_{n-1})(s_1 s_3 s_5 \cdots s_n)$$

provides the required counterexample.

The parity condition on n now forces $X = \widetilde{C}_{n-1}$ for n even, and these graphs are on the list.

We have now reduced to the case where X has at most one edge labelled 4. If no such edge exists, we are in type A, which is on the list, so suppose there is a unique edge labelled 4. We claim that if this edge is not an extremal edge (which would give type B_n) and not adjacent to an extremal edge (which would give type F_n), then X must be the graph of type $\widetilde{F}_5$ shown in Figure 4. If not, we may reduce to the case where $n = 7$ and $m(s_3, s_4) = 4$. In this case, the required counterexample can be taken to be

$$(s_3 s_5 s_7)(s_4 s_6)(s_3 s_5)(s_2 s_4)(s_1 s_3)(s_2 s_4)(s_3 s_5)(s_4 s_6)(s_3 s_5 s_7).$$

Since $\widetilde{F}_5$ is on the list, our proof is complete. $\square$

§7. Concluding remarks

Using the techniques of §2, it is possible to derive sharper results about the structure constants of the c-basis for star reducible Coxeter groups. In particular, writing

$$c_x c_y = \sum_{w \in W_c} f(x, y, w) c_w,$$

one may show that all nonzero Laurent polynomials $f(x, y, w)$, for a fixed x and y, are (positive) integer multiples of the same power of δ.

According to [**4**], interesting algebras and representations defined over $\mathbb{N}$ come from category theory, and are best understood when their categorical origin has been discovered. In [**9**], the author showed how in the case of Coxeter types A, B, H and I, the positivity property of Theorem 4.6 (v) may be understood in terms of a category of tangles. However, there ought to be some representation-theoretic way to understand this, building on the work of Stroppel [**20**, §4] in the case of Coxeter types A, B and D.

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