THE AFFINE q-SCHUR ALGEBRA

R.M. Green

Department of Mathematics and Statistics
Lancaster University
Lancaster LA1 4YF
England
E-mail: r.m.green@lancaster.ac.uk

Abstract. We introduce an analogue of the q-Schur algebra associated to Coxeter systems of type $\widehat{A}_{n-1}$. We give two constructions of this algebra. The first construction realizes the algebra as a certain endomorphism algebra arising from an affine Hecke algebra of type $\widehat{A}_{r-1}$, where $n \geq r$. This generalizes the original q-Schur algebra as defined by Dipper and James, and the new algebra contains the ordinary q-Schur algebra and the affine Hecke algebra as subalgebras. Using this approach we can prove a double centralizer property. The second construction realizes the affine q-Schur algebra as the faithful quotient of the action of a quantum group on the tensor power of a certain module, analogous to the construction of the ordinary q-Schur algebra as a quotient of $U(\mathfrak{gl}_n)$.

Introduction

The q-Schur algebra, $S_q(n,r)$, is a finite dimensional algebra which first appeared in the work of Dipper and James [**5**, **6**] and has applications to the representation theory of the general linear group. The algebra is defined in terms of q-permutation modules for the Hecke algebra arising from the Weyl group of type A. More recently, analogous algebras associated to Weyl groups of type B have been studied (for example, [**15**, **24**]), as well as more sophisticated algebras of the same kind [**7**, **11**] which have applications to representation theory.

An interesting feature of the q-Schur algebra (of type A) is that it arises as a quotient of the quantized enveloping algebra $U(\mathfrak{gl}_n)$. This relationship was explored in [**13**]. It was shown in [**6**] that when $n \geq r$, the q-Schur algebra $S_q(n,r)$ is the centralizing partner of the action of the Hecke algebra $\mathcal{H}(\mathcal{S}_r)$ (where $\mathcal{S}_r$ is the symmetric group on r letters) on "q-tensor space". Jimbo [**18**] established a quantized Weyl reciprocity between the quantized enveloping algebra and the Hecke algebra $\mathcal{H}(\mathcal{S}_r)$ acting on a suitable tensor space. Combining these results means

1991 *Mathematics Subject Classification.* 16S50, 17B37.

The author was supported in part by an E.P.S.R.C. postdoctoral research assistantship.

Typeset by $\mathcal{AMS}$-TEX

that the faithful quotient of this action of the quantized enveloping algebra is the q-Schur algebra.

It is natural to wonder what the corresponding situation is for the Weyl group of type affine A. An affine analogue of Jimbo's quantized Weyl reciprocity has been described by Chari and Pressley [**2**]. The significant difference between the situation considered in [**2**] and the one considered in this paper is that our tensor spaces are infinite dimensional, whereas the one in [**2**] is finite dimensional. We wish to do this so that the tensor space contains the regular representation of our affine Hecke algebra, which is an infinite-dimensional algebra. Since the representation theory of affine Hecke algebras (see [**3**, **21**]) is subtle, our tensor space has an interesting structure as a Hecke algebra module. In particular, it is not completely reducible.

The centralizer algebra of this Hecke algebra action on q-tensor space is the affine q-Schur algebra of the title. This is an infinite dimensional associative algebra which contains the affine Hecke algebra and the finite q-Schur algebra as subalgebras, and like the latter algebra, it is defined in terms of q-permutation modules arising from parabolic subalgebras of a suitable Hecke algebra. We also exhibit two bases of this algebra: firstly a natural one extending the Dipper–James basis of the ordinary q-Schur algebra, and secondly a Kazhdan–Lusztig type basis extending Du's basis [**8**] for the q-Schur algebra.

There is a rival candidate for the title of "affine q-Schur algebra"; this arises (as in the case of the ordinary q-Schur algebra) as a quotient of a suitable Hopf algebra acting on a tensor power $V^{\otimes r}$ of a suitable natural module V. We introduce a quantum group, $U(\widehat{\mathfrak{gl}_n})$, and show that the faithful quotient of the action gives an algebra isomorphic to the affine q-Schur algebra of the preceding paragraph. The tensor power $V^{\otimes r}$ is isomorphic to the q-tensor space of the preceding paragraph, although the isomorphism is less easy to describe than in the finite case. The centralizing algebra of the action $U(\widehat{\mathfrak{gl}_n})$ on $V^{\otimes r}$ is thus isomorphic to the affine Hecke algebra, thus establishing quantized Weyl reciprocity in the affine case.

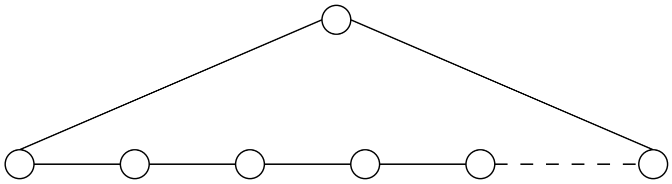
These results raise some interesting questions about canonical bases and other properties of $U(\widehat{\mathfrak{gl}_n})$, which we mention at the end of the paper.

1. Weyl groups and Hecke algebras

1.1 The affine Weyl group.

The Weyl group we consider in this paper is that of type $\widehat{A}_{r-1}$, where we intend $r \geq 3$. This corresponds to the Dynkin diagram in Figure 1.1.

Figure 1.1. Dynkin diagram of type $\widehat{A}_{r-1}$



The number of vertices in the graph in Figure 1.1 is r, as the top vertex (numbered r) is regarded as an extra relative to the remainder of the graph, which is a Coxeter graph of type A_{r-1}.

We associate a Weyl group, $W = W(\widehat{A}_{r-1})$, to this Dynkin diagram in the usual way (as in [**16**, §2.1]). This associates to node i of the graph a generating involution s_i of W, where $s_i s_j = s_j s_i$ if i and j are not connected in the graph, and

$$s_i s_j s_i = s_j s_i s_j$$

if i and j are connected in the graph. For $t \in \mathbb{Z}$, it is convenient to denote by $\bar{t}$ the congruence class of t modulo r, taking values in the set $\{1, 2, \dots, r\}$. For the purposes of this paper, it is helpful to think of this group as follows.

Proposition 1.1.1. *There exists a group isomorphism from W to the set of permutations of $\mathbb{Z}$ which satisfy the following conditions:*

$$(i+r)w = (i)w + r \text{ for } i \in \mathbb{Z}, \tag{1}$$

$$\sum_{t=1}^{r} (t)w = \sum_{t=1}^{r} t \tag{2}$$

such that s_i is mapped to the permutation

$$t \mapsto \begin{cases} t & \text{if } \bar{t} \neq \bar{i}, \overline{i+1}, \\ t-1 & \text{if } \bar{t} = \overline{i+1}, \\ t+1 & \text{if } \bar{t} = \bar{i}, \end{cases}$$

for $t \in \mathbb{Z}$.

Proof. This is given in [**22**]. □

For reasons relating to weight spaces which will become clear later, we consider a larger group $\widehat{W}$ of permutations of $\mathbb{Z}$.

Definition 1.1.2. Let ρ be the permutation of $\mathbb{Z}$ taking t to $t+1$ for all t. Then the group $\widehat{W}$ is defined to be the subgroup of permutations of $\mathbb{Z}$ generated by the group W and ρ.

As will become clear later, the point of ρ is that conjugation by ρ will correspond to a graph automorphism of the Dynkin diagram given by rotation by one place.

Proposition 1.1.3. *There exists a group isomorphism from $\widehat{W}$ to the set of permutations of $\mathbb{Z}$ which satisfy the following conditions:*

$$(i+r)w = (i)w + r \text{ for } i \in \mathbb{Z}, \tag{1}$$

$$\sum_{t=1}^{r} (t)w \equiv \sum_{t=1}^{r} t \mod r. \tag{2}$$

Proof. We first observe that (2) follows from (1). (We have included (2) to emphasise how this result is a generalization of Proposition 1.1.1.)

Next, we note that $\rho s_{i+1} \rho^{-1} = s_i$, where s_{r+1} is interpreted as meaning s_1. This shows that any element of $\widehat{W}$ is expressible as $\rho^z w$ for some integer z and some $w \in W$. Since ρ^z and w both have properties (1) and (2) given in the statement, it follows that $\rho^z w$ does also. Thus any element of $\widehat{W}$ satisfies the properties given.

Conversely, given a permutation w satisfying (1) and (2), condition (2) shows that

$$\sum_{t=1}^{r} (t)w = \left(\sum_{t=1}^{r} t \right) + kr$$

for some integer k. Using the fact that

$$\sum_{t=1}^{r} (t)\rho = \left(\sum_{t=1}^{r} t \right) + r,$$

we see that $w\rho^{-k}$ lies in W by Proposition 1.1.1. It follows that w lies in $\widehat{W}$. □

Corollary 1.1.4. *Any element of $\widehat{W}$ is uniquely expressible in the form $\rho^z w$ for $z \in \mathbb{Z}$ and $w \in W$. Conversely, any element of this form is an element of $\widehat{W}$.*

Proof. The second assertion is obvious. The first assertion follows from the uniqueness of the integer k appearing in the proof of Proposition 1.1.3. □

The group $\widehat{W}$ can be expressed in a more familiar way via a semidirect product.

Proposition 1.1.5. *Let $S \cong \mathcal{S}_r$ be the subgroup of $\widehat{W}$ generated by*

$$\{s_1, s_2, \ldots, s_{r-1}\}.$$

Let Z be the subgroup of $\widehat{W}$ consisting of all permutations z satisfying

$$(t)z \equiv t \mod r$$

for all t. Then $\mathbb{Z}^r \cong Z \lhd \widehat{W}$ and $\widehat{W}$ is the semidirect product of S and Z.

Proof. It is clear that S and Z are isomorphic to the groups given, since any permutation z satisfying the given congruence lies in $\widehat{W}$. Observe also that $S \cap Z = \{1\}$. The other assertions follow easily. □

It is convenient to extend the usual notion of the length of an element of a Coxeter group to the group $\widehat{W}$ in the following way.

Definition 1.1.6. If $w \in W$, the length, $\ell(w)$ of W is the length of a word of minimal length in the group generators s_i of W which is equal to w.

The length, $\ell(w')$, of a typical element $w' = \rho^z w$ of $\widehat{W}$ (where $z \in \mathbb{Z}$ and $w \in W$) is defined to be $\ell(w)$.

1.2 The affine Hecke algebra.

We now define the Hecke algebra $\mathcal{H} = \mathcal{H}(\widehat{W})$ as in [**19**]. (It will be shown in §4.2 that this definition is compatible with the definition given in [**2**].) The Hecke algebra is a q-analogue of the group algebra of $\widehat{W}$, and is related to W in the same way as the Hecke algebra $\mathcal{H}(\mathcal{S}_r)$ of type A is related to the symmetric group $\mathcal{S}_r$. In particular, one can recover the group algebra of $\widehat{W}$ by replacing the parameter q occurring in the definition of $\mathcal{H}(\widehat{W})$ by 1.

Definition 1.2.1. The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\widehat{W})$ over $\mathbb{Z}[q, q^{-1}]$ is the associative, unital algebra with algebra generators

$$\{T_{s_1}, \dots, T_{s_r}\} \cup \{T_\rho, T_\rho^{-1}\}$$

and relations

$$\begin{aligned} T_s^2 &= qT_s + (q-1), && (1)\\ T_sT_t &= T_tT_s \text{ if } s \text{ and } t \text{ are not adjacent in the Dynkin diagram,} && (2)\\ T_sT_tT_s &= T_tT_sT_t \text{ if } s \text{ and } t \text{ are adjacent in the Dynkin diagram,} && (3)\\ T_\rho T_{s_{i+1}} T_\rho^{-1} &= T_{s_i}. && (4) \end{aligned}$$

In relation (4), we interpret s_{r+1} to mean s_1.

Definition 1.2.2. Let $w \in W$. The element T_w of $\mathcal{H}(W)$ is defined as

$$T_{s_{i_1}} \cdots T_{s_{i_m}},$$

where $s_{i_1} \cdots s_{i_m}$ is a reduced expression for w (i.e. one with m minimal). (This is well-defined by standard properties of Coxeter groups.)

If $w' \in \widehat{W}$ is of form $\rho^z w$ for $w \in W$, we denote by $T_{w'}$ the element

$$T_\rho^z T_w.$$

(This is well-defined by Corollary 1.1.4.)

Proposition 1.2.3. *A free $\mathbb{Z}[q, q^{-1}]$-basis for $\mathcal{H}$ is given by the set*

$$\{T_w : w \in \widehat{W}\}.$$

Proof. Clearly, $\mathcal{H}(W)$ is a quotient of the Hecke algebra arising from a Coxeter group of type affine A, and so has a spanning set $\{T_w : w \in W\}$.

Using relation (4) of Definition 1.2.1, one can express any element of $\mathcal{H}$ as a sum of elements $T_\rho^z T$, where T is an element of the subalgebra of $\mathcal{H}(\widehat{W})$ generated by the elements T_s. It follows that the set given in the statement of the proposition is a spanning set.

Suppose that there is a linear dependence relation among the elements in the statement. We invoke a standard argument based on Gauss' Lemma (see [**15**, Theorem 3.2.4]). By clearing denominators and negative powers of q, we may assume all the coefficients in the dependence relation lie in $\mathbb{Z}[q]$ and that the largest monomial involved is of minimal possible degree. If the coefficients become identically zero when q is set to 1, then $(q-1)$ divides each coefficient over $\mathbb{Q}(q)$ and therefore over $\mathbb{Z}[q]$. This contradicts the minimality assumption, so the coefficients are not identically zero when q is set to 1. However, this gives a dependence relation in the group algebra of $\widehat{W}$, a contradiction. $\square$

Remark 1.2.4. The unquantized analogue of the affine Hecke algebra in the sense of [**2**, §4.9] is precisely the semidirect product arising in Proposition 1.1.5. It will turn out in §4.2 that our algebra is the same as the affine Hecke algebra of [**2**].

It is sometimes useful to know that $\mathcal{H}$ can be generated as an algebra more economically as follows.

Lemma 1.2.5. *As a $\mathbb{Z}[q, q^{-1}]$-algebra, $\mathcal{H}$ is generated by T_{s_1}, T_ρ and T_ρ^{-1}.*

Proof. This is obvious from relation (4) of Definition 1.2.1. $\square$

1.3 Graphs.

It is sometimes useful to be able to interpret the length of an element in terms of combinatoric properties of graphs. Any permutation of $\mathbb{Z}$ can be represented by using the points $\mathbb{Z} \times \{0, 1\}$, where point i in the top row is connected to point $(i)w$ in the bottom row by a straight line.

Because of condition (1) in Proposition 1.1.1 and Proposition 1.1.3, the permutations representing elements of $\widehat{W}$ have the property that they can be represented on a cylinder with r points evenly spaced on the top and bottom faces, labelled with the congruence classes 1 up to r modulo r. It is clear that one can pass between the cylindrical and planar notation without loss.

Definition 1.3.1. Let $w \in \widehat{W}$. We define $\nu(w)$ to be the number of crossings in the cylindrical representation of w.

The advantage of representing the permutations on a cylinder can be seen from the following result.

Proposition 1.3.2. *Let $w \in \widehat{W}$. Then $\ell(w) = \nu(w)$.*

Proof. Note that $\nu(s_i) = 1$ and $\nu(\rho^z) = 0$. It follows (by composition of permutations) that $\nu(w) \leq \ell(w)$ in general.

We prove the converse inequality $\nu(w) \geq \ell(w)$ by induction on $\nu(w)$. If $\nu(w) = 0$, one can easily check that $(t)w = t + z$ for some $z \in \mathbb{Z}$, and so $w = \rho^z$.

Now consider the case where the graph of w has a crossing. This means there exist $i < j \in \mathbb{Z}$ with $(i)w > (j)w$. By consideration of the pairs $(i, i+1)$, $(i+1, j-1)$

and $(j-1, j)$, one sees that there exists such a pair i and j for which $j = i+1$. Now the element ws_i of W' is such that $\nu(ws_i) = \nu(w) - 1$. By induction, ws_i is of length at most $\nu(w) - 1$, so $w = ws_is_i$ is of length at most $\nu(w)$, as desired. This completes the induction. □

This interpretation of length leads immediately to the following corollary.

Corollary 1.3.3. *We have*

$$(i)w < (i+1)w \Leftrightarrow \ell(s_iw) > \ell(w)$$

and

$$(i)w^{-1} < (i+1)w^{-1} \Leftrightarrow \ell(ws_i) > \ell(w).$$

1.4 Distinguished coset representatives.

A well-known property of Coxeter groups which is used extensively in the theory of q-Schur algebras is that of distinguished coset representatives with respect to parabolic subgroups. These concepts can be sensibly extended to the group $\widehat{W}$. Note that $\widehat{W}$ is not a Coxeter group in an obvious way, although W is.

Definition 1.4.1. Let Π be the set of subsets of $S = \{s_1, s_2, \dots, s_r\}$, excluding S itself. For each $\pi \in \Pi$, we define the subgroup $\widehat{W}_\pi$ of $\widehat{W}$ to be that generated by $\{s_i \in \pi\}$. (Such a subgroup is called a parabolic subgroup.) We will sometimes write W_π for $\widehat{W}_\pi$ to emphasise that it is a subgroup of W.

Let Π' be the set of elements of Π which omit the generator s_r.

Remark 1.4.2. All the subgroups $\widehat{W}_\pi$ are in fact subgroups of W, and are parabolic subgroups in the usual sense of Coxeter groups. Furthermore, each such $\widehat{W}_\pi$ is isomorphic to a direct product of Coxeter groups of type A (i.e. finite symmetric groups) corresponding to the connected components of the Dynkin diagram obtained after omitting the elements s_i which do not occur in π. This assertion is immediate from well-known properties of Coxeter groups.

Definition 1.4.3. Let $\pi \in \Pi$. The subset $\mathcal{D}_\pi$ of $\widehat{W}$ is the set of those elements such that for any $w \in \widehat{W}_\pi$ and $d \in \mathcal{D}_\pi$,

$$\ell(wd) = \ell(w) + \ell(d).$$

We call $\mathcal{D}_\pi$ the set of distinguished right coset representatives of $\widehat{W}_\pi$ in $\widehat{W}$.

The subset $\mathcal{D}_\pi^{-1}$ is called the set of distinguished left coset representatives of $\widehat{W}_\pi$ in $\widehat{W}$; elements $d \in \mathcal{D}_\pi^{-1}$ have the property that

$$\ell(dw) = \ell(d) + \ell(w)$$

for any $w \in \widehat{W}_\pi$.

Proposition 1.4.4. *Let $\pi \in \Pi$ and $w \in \widehat{W}$. Then $w = w_\pi w^\pi$ for a unique $w_\pi \in \widehat{W}_\pi$ and $w^\pi \in \mathcal{D}_\pi$.*

We have $w \in \mathcal{D}_\pi$ if and only if $(t)w < (t+1)w$ whenever $(t, t+1)$ corresponds to a generator $s_{\bar{t}}$ in $\widehat{W}_\pi$.

Note. This is a familiar fact for Coxeter groups, but needs to be checked in the case of the group $\widehat{W}$.

Proof. The second assertion follows from the permutation representation of $\widehat{W}$ given by Proposition 1.1.3, Corollary 1.3.3, and the fact that $\widehat{W}_\pi$ is canonically isomorphic to a direct product of finite symmetric groups.

Using the same techniques, we see that any $w \in \widehat{W}$ arises as a product $w_\pi w^\pi$ such that the lengths add up as claimed. Furthermore, any element $w' w_\pi$ for $w' \in \widehat{W}_\pi$ satisfies $\ell(w' w_\pi) = \ell(w') + \ell(w_\pi)$. It follows that the set $\mathcal{D}_\pi$ is indeed a set of right coset representatives. The uniqueness of the decomposition also follows. $\square$

It is clear that $\mathcal{D}_\pi^{-1}$ is a set of distinguished left coset representatives with properties analogous to the right coset representatives given by Proposition 1.4.4.

Proposition 1.4.5. *Let $\pi_1, \pi_2 \in \Pi$. The set $\mathcal{D}_{\pi_1, \pi_2} := \mathcal{D}_{\pi_1} \cap \mathcal{D}_{\pi_2}^{-1}$ is an irredundantly described set of double $\widehat{W}_{\pi_1}$–$\widehat{W}_{\pi_2}$-coset representatives, each of minimal length in its double coset.*

Proof. This follows from Proposition 1.4.4 using an elementary argument. $\square$

The double coset representatives $\mathcal{D}_{\lambda,\mu}$ will play a key rôle in the understanding the structure of the affine q-Schur algebra.

2. The affine q-Schur algebra as an endomorphism algebra

2.1 Construction of the affine q-Schur algebra.

Using the affine Hecke algebra of §1, we can now define the affine q-Schur algebra. The definition is designed to be compatible with a quantum group which will be introduced in §3, where the motivation for the definitions here will become clearer.

The following definition was made in [**6**].

Definition 2.1.1. A weight is a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of r into n pieces, that is, a finite sequence of nonnegative integers whose sum is r. (There is no monotonicity assumption on the sequence.) We denote the set of weights by $\Lambda(n, r)$.

The r-tuple $\ell(\lambda)$ of a weight λ is the weakly increasing sequence of integers where there are λ_i occurrences of the entry i.

The Young subgroup $\mathcal{S}_\lambda \subseteq \mathcal{S}_r \subseteq W \subseteq \widehat{W}$ is that subgroup of permutations of the set $\{1, 2, \ldots, r\}$ which leaves invariant the following sets of integers:

$$\{1, 2, \ldots, \lambda_1\}, \{\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2\}, \{\lambda_1 + \lambda_2 + 1, \ldots\}, \ldots.$$

The weight ω is given by the n-tuple

$$(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{0, 0, \ldots, 0}_{n-r}).$$

Remark 2.1.2. The Young subgroup $\mathcal{S}_\lambda \subseteq \mathcal{S}_r$ can be thought of as a group $\widehat{W}_\lambda$ for some $\lambda \in \Pi'$. Note, however, that different compositions λ can give rise to canonically isomorphic groups. Also note that the hypothesis $n \geq r$ is necessary for ω to exist.

Definition 2.1.3. Let $\lambda \in \Pi$. For $t \in \mathbb{Z}$, the parabolic subgroup $\widehat{W}_{\lambda+t}$ is the one generated by those elements $s_{\overline{i+t}}$ where i is such that s_i lies in $\widehat{W}_\lambda$. We also use the notation $\mathcal{D}_{\lambda+t}$ with the obvious meaning.

The element $x_{\lambda+t} \in \mathcal{H}$ is defined as

$$x_{\lambda+t} := \sum_{w \in \widehat{W}_{\lambda+t}} T_w.$$

We will write x_λ for $x_{\lambda+0}$.

Definition 2.1.4. The affine q-Schur algebra $\widehat{S}_q(n,r)$ over $\mathbb{Z}[q,q^{-1}]$ is defined by

$$\widehat{S}_q(n,r) := \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}\right),$$

where $\mathcal{H} = \mathcal{H}(\widehat{W})$.

2.2 A basis for $\widehat{S}_q(n,r)$.

One can describe a basis for $\widehat{S}_q(n,r)$ using techniques familiar from [**4**]. The main problem is to describe a basis for the Hom space $\mathrm{Hom}_{\mathcal{H}}(x_\mu \mathcal{H}, x_\lambda \mathcal{H})$.

Lemma 2.2.1. *Let $\lambda \in \Pi'$. As right $\mathcal{H}(W)$-modules,*

$$x_\lambda \mathcal{H}(\widehat{W}) \cong \bigoplus_{t \in \mathbb{Z}} x_\lambda \rho^t \mathcal{H}(W) \cong \bigoplus_{t \in \mathbb{Z}} x_{\lambda+t} \mathcal{H}(W).$$

Proof. The first isomorphism follows by Proposition 1.2.3 and Corollary 1.1.4. The second isomorphism follows from the definition of $x_{\lambda+t}$. $\square$

Dipper and James remark, at the end of [**4**, §2], that all the results of that section hold for arbitrary finite Coxeter groups. In fact, the results also hold for the infinite Coxeter group W with respect to finite parabolic subgroups, for the same reasons. This proves the following result.

Lemma 2.2.2. *The set $\{\phi^d_{\lambda+t,\mu} : d \in W \cap \mathcal{D}_{\lambda+t,\mu}\}$ is a $\mathbb{Z}[q,q^{-1}]$-basis for*

$$\mathrm{Hom}_{\mathcal{H}(W)}(x_\mu \mathcal{H}(W), x_{\lambda+t}\mathcal{H}(W)),$$

where

$$\phi^d_{\lambda+t,\mu}(x_\mu) := \sum_{d' \in \mathcal{D}_\nu \cap W_\mu} x_{\lambda+t} T_{dd'} = \sum_{w \in W_{\lambda+t} d W_\mu} T_w$$

and ν is the composition of n corresponding to the standard Young subgroup

$$d^{-1} W_{\lambda+t} d \cap W_\mu$$

of W.

Using this we can deduce a basis theorem for the affine q-Schur algebra.

Definition 2.2.3. Let $d \in \widehat{W}$ be an element of $\mathcal{D}_{\lambda,\mu}$. Write $d = \rho^z c$ (as in Corollary 1.1.4) with $c \in W$. Then the element

$$\phi^d_{\lambda,\mu} \in \mathrm{Hom}(x_\mu \mathcal{H}(\widehat{W}), x_\lambda \mathcal{H}(\widehat{W}))$$

is defined as

$$\begin{aligned} \phi^d_{\lambda,\mu}(x_\mu) &:= \sum_{d' \in \mathcal{D}_\nu \cap W_\mu} x_\lambda T_\rho^z T_{cd'} \\ &= \sum_{d' \in \mathcal{D}_\nu \cap W_\mu} T_\rho^z x_{\lambda+z} T_{cd'} = \sum_{w \in W_{\lambda+z} c W_\mu} T_\rho^z T_w = \sum_{w \in W_\lambda d W_\mu} T_w \end{aligned}$$

where ν is the composition of n corresponding to the standard Young subgroup

$$d^{-1} W_{\lambda+z} d \cap W_\mu$$

of W.

Theorem 2.2.4. *A free $\mathbb{Z}[q, q^{-1}]$-basis for $\widehat{S}_q(n, r)$ is given by the set*

$$\{\phi^d_{\lambda,\mu} : \lambda, \mu \in \Lambda(n, r),\ d \in \mathcal{D}_{\lambda,\mu}\}.$$

Proof. The proof reduces to showing that a basis for $\mathrm{Hom}_{\mathcal{H}(\widehat{W})}(x_\mu \mathcal{H}(\widehat{W}), x_\lambda \mathcal{H}(\widehat{W}))$ is given by

$$\{\phi^d_{\lambda,\mu} : d \in \mathcal{D}_{\lambda,\mu}\}$$

Since $x_\mu \mathcal{H}(\widehat{W})$ is generated by x_μ, we can use lemmas 2.2.1 and 2.2.2 to reduce the problem to finding a basis for

$$\mathrm{Hom}_{\mathcal{H}(W)}(x_\mu \mathcal{H}(\widehat{W}), x_\lambda \mathcal{H}(\widehat{W})) = \bigoplus_{t \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{H}(W)}(x_\mu \mathcal{H}(W), x_{\lambda+t} \mathcal{H}(W)).$$

Lemma 2.2.2 gives this as the set

$$\{\phi^{d'}_{\lambda+t,\mu} : d \in W \cap \mathcal{D}_{\lambda+t,\mu}, t \in \mathbb{Z}\}.$$

But since $\phi^{d'}_{\lambda+t,\mu}$ is a homomorphism from x_μ to $x_\lambda T_\rho^t$, $\phi^d_{\lambda,\mu}$ is a homomorphism from x_μ to x_λ (set $d' = w$ and $t = z$). Furthermore there is a natural bijection between elements of $\mathcal{D}_{\lambda,\mu}$ and pairs $(t, W \cap \mathcal{D}_{\lambda+t,\mu})$ for $t \in \mathbb{Z}$ given by sending d to (z, w) where $d = \rho^z w$. The proof follows. $\square$

It is now easy to show that the affine q-Schur algebra contains the affine Hecke algebra and the ordinary q-Schur algebra.

Proposition 2.2.5. *The set of basis elements*

$$\{\phi^d_{\lambda,\mu} : \lambda, \mu \in \Lambda(n,r), d \in \mathcal{S}_r \cap \mathcal{D}_{\lambda,\mu}\}$$

spans a subalgebra of $\widehat{S}_q(n,r)$ canonically isomorphic to the q-Schur algebra $S_q(n,r)$.

The set of basis elements

$$\{\phi^d_{\omega,\omega} : d \in \widehat{W}\}$$

spans a subalgebra canonically isomorphic to the Hecke algebra $\mathcal{H}(\widehat{W})$, where $\phi^d_{\omega,\omega}$ is identified with T_d.

Proof. We recall that the ordinary q-Schur algebra is defined in formally in the same way as $\widehat{S}_q(n,r)$ (see [**6**]), but with $\mathcal{H}$ being the Hecke algebra of the symmetric group $\mathcal{S}_r$. The first assertion is now immediate.

The second assertion follows from the fact that

$$\mathrm{End}_{\mathcal{H}(W)}(x_\omega \mathcal{H}(W)) = \mathrm{End}_{\mathcal{H}(W)}(\mathcal{H}(W)) \cong \mathcal{H}(W),$$

since $\mathcal{H}(W)$ is an associative algebra with 1. $\square$

2.3 q-tensor space and the double centralizer property.

The definition of the q-Schur algebra in §2.2 allows the notion of q-tensor space to be introduced in exactly the same way as for the finite q-Schur algebra in [**6**]. The motivation for the term "tensor" will become clearer in §3.

Definition 2.3.1. Identifying $T_d \in \mathcal{H}(\widehat{W})$ with $\phi^d_{\omega,\omega} \in \widehat{S}_q(n,r)$, we identify the basis element $\phi^d_{\lambda,w}$ of $\widehat{S}_q(n,r)$ with the coset $x_\lambda T_d$. This makes

$$\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H} = \langle \phi^d_{\lambda,w} : \lambda \in \Lambda(n,r), d \in \mathcal{D}_\lambda \rangle$$

into a $\widehat{S}_q(n,r)$–$\mathcal{H}(\widehat{W})$ bimodule, which we call q-tensor space, $E(n,r)$.

Lemma 2.3.2. *As a left $\widehat{S}_q(n,r)$-module, q-tensor space is generated by $\phi^1_{\omega,\omega}$.*

Proof. This is immediate from the fact that $\phi^d_{\lambda,\omega}.\phi^1_{\omega,\omega} = \phi^d_{\lambda,\omega}$. $\square$

Using the notion of q-tensor space, we can prove that the actions of the affine q-Schur algebra and the affine Hecke algebra on q-tensor space are in reciprocity with each other, that is, that each is the centralizing algebra of the other. This is a straightforward generalization of [**6**, Theorem 6.6].

Theorem 2.3.3 (the double centralizer property).

(i) $\mathrm{End}_{\widehat{S}_q(n,r)}(E(n,r)) = \mathcal{H}(\widehat{W})$.

(ii) $\mathrm{End}_{\mathcal{H}(\widehat{W})}(E(n,r)) = \widehat{S}_q(n,r)$.

Proof. The proof of (ii) follows easily from Definition 2.3.1 and the definition of $\widehat{S}_q(n,r)$.

To prove (i), first observe that Lemma 2.3.2 implies that any endomorphism in $\mathrm{End}_{\widehat{S}_q(n,r)}(E(n,r))$ is determined by its effect on the element x_ω in the subspace $x_\omega \mathcal{H}$ of q-tensor space. Since

$$\phi^1_{\lambda,\lambda}\phi^d_{\mu,\omega} = \delta_{\lambda,\mu}\phi^d_{\mu,w},$$

this endomorphism must map $x_\omega \mathcal{H}$ to itself.

The proof of Proposition 2.2.5 shows that these endomorphisms are in canonical correspondence with elements of $\mathcal{H}(W)$ acting by right multiplication. □

2.4 The Kazhdan–Lusztig basis.

Using the methods of Du [**8**], we can describe a second basis for the algebra $\widehat{S}_q(n,r)$, analogous to the Kazhdan–Lusztig basis for the Hecke algebra of a Coxeter group [**20**]. Such a basis is called an IC (intersection cohomology) basis; this terminology is justified in the case of the ordinary q-Schur algebra, where the basis has intimate connections to Lusztig's canonical basis for U^+ (the "plus part" of a quantized enveloping algebra) and the theory of perverse sheaves (see [**14**, Corollary 4.7]).

The following result is well known for Coxeter groups and their finite parabolic subgroups (each of which has a unique longest element).

Lemma 2.4.1. *Let $\lambda, \mu \in \Pi$ and $w \in \widehat{W}$. The double coset $\widehat{W}_\lambda w \widehat{W}_\mu$ has a unique element of maximal length, denoted by w^+.*

Proof. We may assume that $w \in \mathcal{D}_{\lambda,\mu}$. Write $w = \rho^t w'$ for $w' \in W$. Then

$$\widehat{W}_\lambda w \widehat{W}_\mu = \rho^t \widehat{W}_{\lambda+t} w' \widehat{W}_\mu.$$

Using the result for Coxeter groups, we know that $\widehat{W}_{\lambda+t} w' \widehat{W}_\mu$ has a longest element, w'^+. It is easy to check that the element $w^+ = \rho^t w'^+$ satisfies the required criteria. □

Definition 2.4.2. We denote by $\leq$ the (strong) Bruhat order on a Coxeter group (see [**16**, §5.9] for the definition). We extend this order to the group $\widehat{W}$ by stipulating that $\rho^a y \leq \rho^b w$ (for $y, w \in W$) if and only if $a = b$ and $y \leq w$.

Let $P_{y,w}$ be the Kazhdan–Lusztig polynomial for $y, w \in W$. This polynomial is guaranteed to be zero unless $y \leq w$. We extend these polynomials to the group W by defining $P_{\rho^a y, \rho^b w} := \delta_{a,b} P_{y,w}$.

The indeterminate v is defined to be $q^{1/2}$.

The element $w_{0,\mu}$ is the longest element of $\widehat{W}_\mu$.

We can now define the Kazhdan–Lusztig basis of $\widehat{S}_q(n,r)$, taken over the base ring $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$.

Definition 2.4.3. Let $\{\phi^d_{\lambda,\mu}\}$ be the basis of $\widehat{S}_q(n,r)$ defined in §2.2. We define

$$\theta^d_{\lambda,\mu} := v^{\ell(w_{0,\mu})} \sum_{z \in \mathcal{D}_{\lambda,\mu}} \alpha_{z,w} \phi^d_{\lambda,\mu},$$

where

$$\alpha_{z,w} := v^{-\ell(w^+)} P_{z^+,w^+}.$$

Remark 2.4.4. Note that the sum occurring above is a finite sum, because for any $w \in \widehat{W}$, there is only a finite number of elements y such that $y \leq w$.

It is clear that the basis $\{\theta^d_{\lambda,\mu}\}$ is a free $\mathcal{A}$-basis for $\widehat{S}_q(n,r)$ since the basis change matrix from the basis $\{\phi^d_{\lambda,\mu}\}$ is a unit multiple of a unitriangular matrix with respect to a suitable ordering.

2.5 Generators and relations.

For the purposes of §3, it is convenient to have a presentation of $\widehat{S}_q(n,r)$ over $\mathbb{Q}(v)$ via generators and relations.

Proposition 2.5.1. *The algebra $\mathbb{Q}(v) \otimes \widehat{S}_q(n,r)$ is generated by elements*

$$\{\phi^d_{\omega,\omega} : d \in \widehat{W}\} \cup \{\phi^1_{\lambda,\omega} : \lambda \in \Lambda(n,r)\} \cup \{\phi^1_{\omega,\lambda} : \lambda \in \Lambda(n,r)\}.$$

The elements $\phi^d_{\omega,\omega}$ are subject to the relations of the affine Hecke algebra of Definition 1.2.1 under the identification given by Proposition 2.2.5. The generators are also subject to the following relations, where s denotes a generator $s_i \in \widehat{W}_\lambda$.

$$\phi^1_{\omega,\lambda}\phi^1_{\mu,\omega} = \delta_{\lambda,\mu} \sum_{d \in \widehat{W}_\lambda} \phi^d_{\omega,\omega}, \tag{1}$$

$$\phi^s_{\omega,\omega}\phi^1_{\omega,\lambda} = q\phi^1_{\omega,\lambda}, \tag{2}$$

$$\phi^1_{\lambda,\omega}\phi^s_{\omega,\omega} = q\phi^1_{\lambda,\omega}. \tag{3}$$

Note. Note that $s \neq s_r$ above, since $\lambda \in \Pi'$. □

Proof. Using Proposition 2.2.5 and properties of the ordinary q-Schur algebra (or the definition of the ϕ-basis), it can be verified that the relations given are all true in $\widehat{S}_q(n,r)$.

Let $d = \rho^z c \in \mathcal{D}_{\lambda,\mu}$, where $c \in W$ (and thus $c \in \mathcal{D}_{\lambda+z,\mu}$). It follows from Definition 2.2.3 that

$$\begin{aligned}
\phi^1_{\lambda,\omega}\phi^d_{\omega,\omega}\phi^1_{\omega,\mu}(x_\mu) &= T^z_\rho x_{\lambda+z} T_c x_\mu \\
&= \left(\sum_{d' \in \mathcal{D}_\nu \cap W_\mu} T^z_\rho x_{\lambda+z} T_{cd'} \right) x_\nu \\
&= P_\nu(q) \sum_{d' \in \mathcal{D}_\nu \cap W_\mu} T^z_\rho x_{\lambda+z} T_{cd'} \\
&= P_\nu(q).\phi^d_{\lambda,\mu},
\end{aligned}$$

where ν is as in Definition 2.2.3 and P_ν is the Poincaré polynomial

$$P_\nu(q) := \sum_{w \in \widehat{W}_\nu} q^{\ell(w)}.$$

This relies on the fact that the expression in parentheses in the second line corresponds to a union of left cosets of $\widehat{W}_\mu$, and $\widehat{W}_\nu \subseteq \widehat{W}_\mu$.

It follows that the given set does generate $\widehat{S}_q(n, r)$, since $P_\nu(q)$ is invertible. It also follows that the relations given are sufficient to express the product of two basis elements as a linear combination of others, which completes the proof. $\square$

3. The affine q-Schur algebra as a Hopf algebra quotient

The aim of §3 is to realize the affine q-Schur algebra as the faithful quotient of the action of a certain quantum group acting on a tensor power of a natural module. This explains why the definition of the q-Schur algebra as given in §2 is a natural one. The tensor power of the natural module will be seen to be the analogue of the q-tensor space of §2, thus justifying the terminology.

In this section, we take the base ring of $\widehat{S}_q(n, r)$ and $\mathcal{H}(\widehat{W})$ to be $\mathbb{Q}(v)$ by tensoring in the obvious way.

3.1 The Hopf algebra $U(\widehat{\mathfrak{gl}_n})$.

We now introduce a quantum group, $U(\widehat{\mathfrak{gl}_n})$, based on the quantized enveloping algebra $U(\widehat{\mathfrak{sl}_n})$ associated to a Dynkin diagram of type $\widehat{A}_{n-1}$. From now on, $^-$ will denote congruence modulo n (as opposed to modulo r), unless otherwise specified.

The main differences between $U(\widehat{\mathfrak{gl}_n})$ and $U(\widehat{\mathfrak{sl}_n})$ are that the element K_i in $U(\widehat{\mathfrak{sl}_n})$ corresponds to the element $K_i K_{\overline{i+1}}^{-1}$ in $U(\widehat{\mathfrak{gl}_n})$ (similar to the difference between $U(\mathfrak{gl}_n)$ and $U(\mathfrak{sl}_n)$), and, more strikingly, $U(\widehat{\mathfrak{gl}_n})$ contains a new grouplike element, R, which will play an important rôle in the results to come.

Definition 3.1.1. The associative, unital algebra $U(\widehat{\mathfrak{gl}_n})$ over $\mathbb{Q}(v)$ is given by generators

$$E_i, F_i, K_i, K_i^{-1}, R, R^{-1}$$

(where $1 \leq i \leq n$) subject to the following relations:

$$
\begin{aligned}
&K_iK_j = K_jK_i, && (1)\\
&K_iK_i^{-1} = K_i^{-1}K_i = 1, && (2)\\
&K_iE_j = v^{\epsilon^+(i,j)}E_jK_i, && (3)\\
&K_iF_j = v^{\epsilon^-(i,j)}F_jK_i, && (4)\\
&E_iF_j - F_jE_i = \delta_{ij}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{v - v^{-1}}, && (5)\\
&E_iE_j = E_jE_i \quad \text{if } i \text{ and } j \text{ are not adjacent}, && (6)\\
&F_iF_j = F_jF_i \quad \text{if } i \text{ and } j \text{ are not adjacent}, && (7)\\
&E_i^2E_j - (v + v^{-1})E_iE_jE_i + E_jE_i^2 = 0 \quad \text{if } i \text{ and } j \text{ are adjacent}, && (8)\\
&F_j^2F_i - (v + v^{-1})F_jF_iF_j + F_iF_j^2 = 0 \quad \text{if } i \text{ and } j \text{ are adjacent}, && (9)\\
&RR^{-1} = R^{-1}R = 1, && (10)\\
&R^{-1}K_{\overline{i+1}}R = K_i, && (11)\\
&R^{-1}K_{\overline{i+1}}^{-1}R = K_i^{-1}, && (12)\\
&R^{-1}E_{\overline{i+1}}R = E_i, && (13)\\
&R^{-1}F_{\overline{i+1}}R = F_i. && (14)
\end{aligned}
$$

By "adjacent", we mean that i and j index adjacent nodes in the Dynkin diagram. Here,

$$\epsilon^+(i,j) := \begin{cases} 1 & \text{if } j = i; \\ -1 & \text{if } \bar{j} = \overline{i-1}; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\epsilon^-(i,j) := \begin{cases} 1 & \text{if } \bar{j} = \overline{i-1}; \\ -1 & \text{if } j = i; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.1.2. Although it appears at first that the set of defining relations is rather large, it should be noted that in fact the algebra may be generated using only $E_1, F_1, K_1, K_1^{-1}, R, R^{-1}$, since the other generators may be expressed in terms of these by using relations (11)–(14). (Compare this with Lemma 1.2.5.) Also note that R^n lies in the centre of $U(\widehat{\mathfrak{gl}_n})$, just as T_ρ^r lies in the centre of $\mathcal{H}(\widehat{W})$.

It will turn out that the algebra $U(\widehat{\mathfrak{gl}_n})$ can be given the structure of a Hopf algebra (which justifies the use of the term "quantum group"). This is an extension of the well-known Hopf algebra structure [**17**, §4.8] of the quantized enveloping algebras associated to Kac–Moody Lie algebras. Since many of the arguments are familiar, we will highlight only the differences between the situation of $U = U(\widehat{\mathfrak{gl}_n})$ and quantized enveloping algebras. Although we do not need the antipode in the Hopf algebra for our purposes, we introduce it for completeness.

Lemma 3.1.3. *There is a unique homomorphism of unital $\mathbb{Q}(v)$-algebras $\Delta : U \to U \otimes U$ such that*

$$\begin{aligned}
\Delta(1) &= 1 \otimes 1, \\
\Delta(E_i) &= E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i, \\
\Delta(F_i) &= K_i^{-1} K_{i+1} \otimes F_i + F_i \otimes 1, \\
\Delta(X) &= X \otimes X \text{ for } X \in \{K_i, K_i^{-1}, R, R^{-1}\}.
\end{aligned}$$

Note. The elements X above are known as grouplike elements.

Proof. This follows by a routine check, which is familiar except for the cases involving the elements R and R^{-1}. $\square$

Lemma 3.1.4. *The map Δ is coassociative. In other words, $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$.*

Proof. Since Δ is an algebra homomorphism, it suffices to check this on generators. The only unfamiliar case involves R and R^{-1}; in this case, both sides send R to $R \otimes R \otimes R$ and R^{-1} to $R^{-1} \otimes R^{-1} \otimes R^{-1}$. $\square$

Lemma 3.1.5. *There is a unique homomorphism $\varepsilon : U \to \mathbb{Q}(v)$ of unital $\mathbb{Q}(v)$-algebras such that*

$$\varepsilon(E_i) = \varepsilon(F_i) = 0$$

and

$$\varepsilon(K_i) = \varepsilon(K_i^{-1}) = \varepsilon(R) = \varepsilon(R^{-1}) = 1.$$

Proof. It is trivial to check that ε preserves all the relations. $\square$

Lemma 3.1.6. *For any $u \in U$, we have*

$$(\varepsilon \otimes 1)\Delta(u) = 1 \otimes u$$

and

$$(1 \otimes \varepsilon)\Delta(u) = u \otimes 1.$$

Proof. It is enough to verify these claims on the generators. This is easy (including the new cases of $u \in \{R, R^{-1}\}$). $\square$

Lemma 3.1.7. *There is a $\mathbb{Q}(v)$-linear map $S : U \to U$ uniquely determined by the following properties:*

$$\begin{aligned}
S(E_i) &= -E_i K_i^{-1} K_{i+1}, \\
S(F_i) &= -K_i K_{i+1}^{-1} F_i, \\
S(K_i) &= K_i^{-1}, \\
S(K_i^{-1}) &= K_i, \\
S(R) &= R^{-1}, \\
S(R^{-1}) &= R, \\
\forall a, b \in U, \ S(ab) &= S(b)S(a).
\end{aligned}$$

Proof. We verify that S preserves the relations of U, from which it follows that it is determined by its effect on the generators. The only new relations to be checked are relation (10) of Definition 3.1.1 (which is clear) and relations (11)–(14), which are of the form

$$S(R^{-1}X_{i+1}R) = S(X_i)$$

for $X_i \in \{E_i, F_i, K_i, K_i^{-1}\}$. Expanding the left hand side, we obtain

$$S(R^{-1}X_{i+1}R) = S(R)S(X_{i+1})S(R^{-1}) = R^{-1}S(X_{i+1})R.$$

It is then easily verified that $R^{-1}S(X_{i+1})R = S(X_i)$ in each case. □

Definition 3.1.8. The $\mathbb{Q}(v)$-algebra homomorphism $\eta : \mathbb{Q}(v) \to U$ is defined to send $1 \in \mathbb{Q}(v)$ to $1 \in U$.

We denote the multiplication map in U by μ.

Lemma 3.1.9. *We have*

$$\mu(S \otimes 1)\Delta = \eta\varepsilon$$

and

$$\mu(1 \otimes S)\Delta = \eta\varepsilon.$$

Proof. We first check these assertions on the generators. The only new cases involve the elements R and R^{-1}, which involve straightforward checks such as

$$\mu(S \otimes 1)\Delta(R) = \mu(S \otimes 1)(R \otimes R) = \mu(R^{-1} \otimes R) = 1 = \eta(1) = \eta\varepsilon(R).$$

This does not complete the proof, because the map S is not an algebra homomorphism, but there is a standard argument (given in [**17**, §3.7]) which shows that if the relations in the statement hold on the generators, they hold in general. □

Theorem 3.1.10. *The algebra $U(\widehat{\mathfrak{gl}_n})$ is a Hopf algebra with multiplication μ, unit η, comultiplication Δ, counit ε and antipode S.*

Proof. This is immediate from lemmas 3.1.3, 3.1.4, 3.1.5, 3.1.6, 3.1.7 and 3.1.9. □

3.2 Tensor space.

Let V be the $\mathbb{Q}(v)$-vector space with basis $\{e_t : t \in \mathbb{Z}\}$. This has a natural $U(\widehat{\mathfrak{gl}_n})$-module structure as follows.

Lemma 3.2.1. *There is a left action of $U(\widehat{\mathfrak{gl}_n})$ on V defined by the conditions*

$$\begin{aligned}
E_i e_{t+1} &= e_t \text{ if } i = t \mod n,\\
E_i e_{t+1} &= 0 \text{ if } i \neq t \mod n,\\
F_i e_t &= e_{t+1} \text{ if } i = t \mod n,\\
F_i e_t &= 0 \text{ if } i \neq t \mod n,\\
K_i e_t &= v e_t \text{ if } i = t \mod n,\\
K_i e_t &= e_t \text{ if } i \neq t \mod n,\\
R e_t &= e_{t+1}.
\end{aligned}$$

Proof. The actions of K_i^{-1} and R^{-1} are clear from the above conditions and relations (2) and (10) of Definition 3.1.1. One then checks that the other relations are preserved. $\square$

Since $U(\widehat{\mathfrak{gl}_n})$ is a Hopf algebra, the tensor product of two $U(\widehat{\mathfrak{gl}_n})$-modules has a natural $U(\widehat{\mathfrak{gl}_n})$-module structure via the comultiplication Δ.

Definition 3.2.2. The vector space $V^{\otimes r}$ has a natural $U(\widehat{\mathfrak{gl}_n})$-module structure given by $u.v = \Delta(u)^{(r-1)}.v$. We call this module tensor space.

Remark 3.2.3. The definition of $\widehat{S}_q(n,r)$ in §2 is chosen so that the resulting q-tensor space is compatible with the tensor space above. This relationship will be explored later.

It is convenient to introduce certain elements of $\mathrm{End}_U(V^{\otimes r})$, which we call y_i since they are reminiscent of elements of the Hecke algebra in the sense of [**2**, §3.1]. This similarity will be explored in §4.2.

Definition 3.2.4. For each $1 \leq t \leq r$, we define the invertible linear map y_t on $V^{\otimes r}$ by the condition

$$(e_{j_1} \otimes \cdots \otimes e_{j_r}).y_t := e_{j_1} \otimes \cdots \otimes e_{j_{t-1}} \otimes e_{-n+j_t} \otimes e_{j_{t+1}} \otimes \cdots \otimes e_{j_r}.$$

We denote the subspace of V spanned by the elements $\{e_1, \ldots, e_n\}$ by V_n. (This is the vector space which plays the rôle of V in the case of the ordinary q-Schur algebra.) The following result is a useful tool for reducing problems about $\widehat{S}_q(n,r)$ to that of the finite q-Schur algebra.

Proposition 3.2.5. *The elements y_i, y_i^{-1} lie in $\mathrm{End}_U(V^{\otimes r})$. Therefore the action of an element $u \in U(\widehat{\mathfrak{gl}_n})$ on tensor space is determined by its action on $V_n^{\otimes r}$.*

Proof. Since R^n lies in the centre of $U(\widehat{\mathfrak{gl}_n})$, the linear maps which send e_t to $e_{t\pm n}$ lie in $\mathrm{End}_U(V)$. Using the comultiplication, this result can be extended to $\mathrm{End}_U(V^{\otimes r})$, proving the first assertion.

The second assertion holds because $V^{\otimes r} = V_n^{\otimes r}.Y$, where Y is the algebra of endomorphisms of $V^{\otimes r}$ generated by the y_i and their inverses. $\square$

We require the concept of a weight space of tensor space for the results which follow. This is a straightforward generalization of the situation in finite type A.

Definition 3.2.6. The weight $\lambda \in \Lambda(n,r)$ of a basis element

$$e_{t_1} \otimes e_{t_2} \otimes \cdots \otimes e_{t_r}$$

of $V^{\otimes r}$ is given by the condition

$$\lambda_i := |\{j : t_j \equiv i \mod n\}|.$$

The λ-weight space, V_λ, of $V^{\otimes r}$ is the span of all the basis vectors of weight λ.

3.3 Relationship between $U(\widehat{\mathfrak{gl}_n})$ and $\mathcal{H}(\widehat{W})$.

As one might expect from the situation in §2, the weight space V_ω is of particular importance. In this section we define certain operations on this space arising from the action of elements of $U(\widehat{\mathfrak{gl}_n})$. The motivation for doing this is that these operations will generate an algebra isomorphic to $\mathcal{H}(\widehat{W})$. This will provide a link with the results of §2.

Definition 3.3.1. For each $1 \leq i \leq r$, let $\tau(T_{s_i}) : V_\omega \to V_\omega$ be the endomorphism corresponding to the action of $vF_iE_i - 1 \in U(\widehat{\mathfrak{gl}_n})$. Similarly, let $\tau(T_{\rho^{-1}})$ be the endomorphism corresponding to

$$F_nF_{n-1}\cdots F_{r+1}R,$$

and let $\tau(T_\rho)$ be the endomorphism corresponding to

$$E_rE_{r+1}\cdots E_{n-1}R^{-1}.$$

Lemma 3.3.2. *The endomorphisms $\tau(T_w)$ defined above satisfy the relations of Definition 1.2.1 (after replacing T_w by $\tau(T_w)$).*

Proof. Using the epimorphism $\alpha_r : U(\mathfrak{gl}_n) \twoheadrightarrow S_q(n,r)$ described in [**13**], one finds that the action of $\tau(T_{s_i})$ on V_ω in the case where $i \neq r$ corresponds to the action of $\phi^{s_i}_{\omega,\omega} \in S_q(n,r)$. (Recall from [**13**] that $S_q(n,r)$ is the quotient of $U(\mathfrak{gl}_n)$ by the annihilator of $V_n^{\otimes r}$.) This proves relations (1)–(3) of Definition 1.2.1 in the case where s_r is not involved.

It suffices now to prove relation (4), from which the other cases of relations (1)–(3) may be deduced. The effect of $\tau(T_\rho)$ on V_ω is

$$\tau(T_\rho)(e_{i_1} \otimes \cdots \otimes e_{i_r}) = e_{j_1} \otimes \cdots \otimes e_{j_1},$$

where $j_t = i_t - 1 \mod r$. The effect of $\tau(T_{\rho^{-1}})$ on V_ω is the inverse of this action. The proof of relation (4) now follows by calculation of the action of $vF_iE_i - 1$ on V_ω using the comultiplication. $\square$

In the following result, we temporarily replace the v occurring in the definition of the action of $\tau(T_w)$ by 1, in order to regard the endomorphisms $\tau(T_w)$ as endomorphisms of a version of tensor space over $\mathbb{Q}$. This modified tensor space is the r-fold tensor power of the $\mathbb{Q}$-vector space with basis $\{e_t : t \in \mathbb{Z}\}$. We call this process specializing v to 1.

Lemma 3.3.3. *Specialize the parameter v to 1. Then the map taking $\tau(T_w)$ to w extends uniquely to an isomorphism of algebras between the group algebra $\mathbb{Q}\widehat{W}$ of $\widehat{W}$ and the algebra $\tau(\mathcal{H})$ generated by the endomorphisms $\tau(T_w)$ of Definition 3.3.1.*

Proof. It follows from Lemma 3.3.2 that the algebra generated by the $\tau(T_w)$ is a quotient of the group algebra of $\widehat{W}$.

To complete the proof, it is sufficient (by Proposition 1.1.3) to show that $w \in \widehat{W}$ corresponds under the isomorphism given to an endomorphism $\tau \in \tau(\mathcal{H})$ such that

$$\tau(e_{i_1} \otimes \cdots \otimes e_{i_r}) = e_{w(i_1)} \otimes \cdots \otimes e_{w(i_r)}.$$

(The left action of $\widehat{W}$ is defined by $\rho(t) = t - 1$ and s_1 acting via the simple transposition $(1, 2)$. It follows from Proposition 1.1.3 that this is faithful.)

This claim is easily checked when w is one of the generators s_1, ρ or ρ^{-1} of $\widehat{W}$. The general case follows by analysing the action of $\tau(T_w)$ on V_ω in the case $v = 1$. A check shows that this is given by

$$\tau(T_w)(e_{i_1} \otimes \cdots \otimes e_{i_r}) = e_{w(i_1)} \otimes \cdots \otimes e_{w(i_r)}.$$

The proof follows from this. □

We can obtain an analogue of this result which holds in the quantized case.

Lemma 3.3.4. *The map taking $\tau(T_w)$ to T_w extends uniquely to an isomorphism of algebras between $\mathcal{H}(\widehat{W})$ and the algebra $\tau(\mathcal{H})$ generated by the endomorphisms $\tau(T_w)$ of Definition 3.3.1.*

Proof. Lemma 3.3.2 shows that the elements $\tau(T_w)$ satisfy the required defining relations, so that $\tau(T_w)$ is a quotient of $\mathcal{H}(\widehat{W})$.

Suppose an element of $\mathcal{H}(\widehat{W})$ maps to zero under this quotient map. Then by the usual technique of clearing denominators and dividing out by $v - 1$, we can find an element of $\mathbb{Q}\widehat{W}$ which maps to zero under the isomorphism of Lemma 3.3.3. This completes the proof. □

3.4 The map $\alpha_r : U(\widehat{\mathfrak{gl}_n}) \twoheadrightarrow \widehat{S}_q(n, r)$.

The results of §3.3 can be extended to the whole tensor space $V^{\otimes r}$ by using the generators and relations for $\widehat{S}_q(n, r)$ which were given in §2.5. This construction relies on the surjective homomorphism α_r from $U(\mathfrak{gl}_n)$ to $S_q(n, r)$ which was studied in [**13**].

Definition 3.4.1. For each $\lambda \in \Lambda(n, r)$, choose

$$\begin{aligned} E(\lambda) &\in \alpha_r^{-1}(\phi^1_{\lambda,\omega}), \\ F(\lambda) &\in \alpha_r^{-1}(\phi^1_{\omega,\lambda}), \\ G(\lambda) &\in \alpha_r^{-1}(\phi^1_{\lambda,\lambda}). \end{aligned}$$

We also denote by $E(\lambda)$ (respectively, $F(\lambda)$, $G(\lambda)$) the elements of $U(\widehat{\mathfrak{gl}_n})$ which are the images of $E(\lambda)$ (respectively, $F(\lambda)$, $G(\lambda)$) under the obvious homomorphism from $U(\mathfrak{gl}_n)$ to $U(\widehat{\mathfrak{gl}_n})$.

Denote by γ the map from $U(\widehat{\mathfrak{gl}_n})$ to $\mathrm{End}(V^{\otimes r})$ corresponding to the action on tensor space arising from the comultiplication.

Lemma 3.4.2. *There is a unique homomorphism of algebras*

$$\kappa : \widehat{S}_q(n,r) \to \mathrm{End}(V^{\otimes r})$$

such that for all $\lambda \in \Lambda(n,r)$,

$$\begin{aligned}
\kappa(\phi^1_{\lambda,\omega}) &= \gamma(E(\lambda)),\\
\kappa(\phi^1_{\omega,\lambda}) &= \gamma(F(\lambda)),\\
\kappa(\phi^1_{\lambda,\lambda}) &= \gamma(G(\lambda)),\\
\kappa(\phi^{s_1}_{\omega,\omega}) &= \tau(T_{s_1})\gamma(G(\omega)),\\
\kappa(\phi^{\rho^{\pm 1}}_{\omega,\omega}) &= \tau(T_{\rho^{\pm 1}})\gamma(G(\omega)).
\end{aligned}$$

Proof. If such a homomorphism exists, it is unique by Lemma 1.2.5 and Proposition 2.5.1.

It remains to check the relations of Proposition 2.5.1. The Hecke algebra style relations hold because of Lemma 3.3.4. The other relations are essentially relations of the ordinary q-Schur algebra. These hold because of Proposition 3.2.5. $\square$

Lemma 3.4.3. *Let* $w \in \mathcal{S}_r$, $\lambda \in \Lambda(n,r)$, *and let* $w = w_\lambda d$ *be the decomposition of* w *as in Proposition 1.4.4, so that* $d \in \mathcal{D}_\lambda$. *Consider the basis element*

$$e_{i_1} \otimes \cdots \otimes e_{i_r} = e_{(1)w^{-1}} \otimes \cdots \otimes e_{(r)w^{-1}}$$

of $V_n^{\otimes r} \subset V^{\otimes r}$. *Then the action of* $E(\lambda)$ *sends this element to one of the form*

$$v^{2\ell(w_\lambda)+f(\lambda)} e_{j_1} \otimes \cdots \otimes e_{j_r},$$

where j_t *is the* i_t*-th component of* $\ell(\lambda)$ *as in Definition 2.1.1, and* f *is a certain integer-valued function.*

Proof. This is essentially a statement about the ordinary q-Schur algebra, and is therefore a consequence of [**12**, Theorem 3.8]. Alternatively, one can argue as follows.

The claim holds in the case $w = 1$ using the definition of q-tensor space (see [**6**, Definition 2.6] and the discussion following it), and the fact that q-tensor space and tensor space are canonically isomorphic (in the finite dimensional case) by multiplication by a suitable integer power of v (see [**9**, §1.5]). Note that this canonical isomorphism respects the notion of weight space (see Definition 2.1.1 and Definition 3.2.6).

The general case uses the fact that the actions of $\mathcal{H}(\mathcal{S}_r)$ and $U(\mathfrak{gl}_n)$ on $V_n^{\otimes r}$ commute, and the definition of the action of $\mathcal{H}(\mathcal{S}_r)$ as given in [**9**]. $\square$

Remark 3.4.4. This result determines the action of $E(\lambda)$ on V_ω because of Proposition 3.2.5.

Lemma 3.4.5. *Consider the basis element*

$$e_{\ell(\mu)_1} \otimes \cdots \otimes e_{\ell(\mu)_r}$$

of $V_n^{\otimes r} \subset V^{\otimes r}$. *Then the action of* $F(\mu)$ *sends this element to one of the form*

$$v^{g(\mu)} \sum_{w \in \widehat{W}_\mu} v^{\ell(w)} e_{(1)w^{-1}} \otimes \cdots \otimes e_{(r)w^{-1}},$$

where g *is a certain integer-valued function.*

Proof. This is proved using the same techniques as in the proof of Lemma 3.4.3, although this is an easier case. $\square$

Lemma 3.4.6. *We have* $\gamma(U) = \mathrm{im}(\kappa)$. *Thus* $\mathrm{im}(\kappa)$ *is a quotient of* U *by the kernel of its action on tensor space.*

Proof. The second assertion follows if we can establish $\gamma(U) \subseteq \mathrm{im}(\kappa)$, because $\kappa(\phi)$ gives the same endomorphism as the action of a suitable element of $U(\widehat{\mathfrak{gl}_n})$ for ϕ a generator of $\widehat{S}_q(n,r)$.

For the first assertion, it is enough to prove that $\gamma(u) \in \mathrm{im}(\kappa)$ for

$$u \in \{E_1, F_1, K_1, K_1^{-1}, R, R^{-1}\}.$$

The case where $u \in \{E_1, F_1, K_1, K_1^{-1}\}$ holds because of the existence of the surjection α_r in the finite case and Proposition 3.2.5.

We now show that $\gamma(R) \in \mathrm{im}(\kappa)$. It is enough to show that $\gamma(R)\gamma(G(\mu)) \in \mathrm{im}(\kappa)$ for all $\mu \in \Lambda(n,r)$, using the decomposition of the identity of $\widehat{S}_q(n,r)$ into orthogonal idempotents $\phi^1_{\mu,\mu}$. Lemma 3.4.3 and Remark 3.4.4 show that the element $\phi^1_{\lambda,\omega}\phi^w_{\omega,\omega}\phi^1_{\omega,\mu}$ maps under κ to a multiple of $\gamma(R)\gamma(G(\mu))$, where $w = \rho^{-\mu_n}$ and where λ is such that R sends the weight space μ to the weight space λ. The case for R^{-1} is similar but uses $w = \rho^{+\mu_1}$. $\square$

The next result also makes use of specialization. There is no problem in replacing v by 1 here since we are working over an integral form where $v - 1$ cannot occur as a factor of a denominator.

Lemma 3.4.7. *The map* κ, *restricted to the* $\mathbb{Z}[q, q^{-1}]$*-form of* $\widehat{S}_q(n,r)$ *given in Theorem 2.2.4, is a monomorphism when the parameter* v *is specialized to* 1.

Proof. Using an easy weight space argument, this can be reduced to showing that the images of basis elements $\phi^d_{\lambda,\mu}$, for fixed λ and μ, are independent.

Suppose

$$k = \kappa\left(\sum_{d \in \mathcal{D}_{\lambda,\mu}} c_d \phi^d_{\lambda,\mu}\right) = 0.$$

This means that k acts as zero on

$$e_\mu := e_{\ell(\mu)_1} \otimes \cdots \otimes e_{\ell(\mu)_r}.$$

Consider the action of one $\phi^d_{\lambda,\mu}$ which appears in the sum on e_μ. By using the factorisation of $\phi^d_{\lambda,\mu}$ as in the proof of Proposition 2.5.1 together with lemmas 3.4.3 and 3.4.5 (in the case $v = 1$), the isomorphism of Lemma 3.3.3 and the definition of κ in Lemma 3.4.2, we have

$$\phi^d_{\lambda,\mu}.e_\mu = \sum_{w \in d\widehat{W}_\mu} l_w e_{\lambda(w(1))} \otimes \cdots \otimes e_{\lambda(w(r))},$$

where $\lambda(i) := \ell(\lambda)_i$, and the scalars l_w are positive integers. It follows from this that the actions of $\phi^d_{\lambda,\mu}$ and $\phi^{d'}_{\lambda,\mu}$ on e_μ (where d, d' lie in distinct double $\widehat{W}_\lambda$–$\widehat{W}_\mu$-cosets) are nonzero and have distinct supports. The claim follows. $\square$

We are now ready to show that the two definitions of the affine q-Schur algebra are compatible with each other.

Theorem 3.4.8. *Over $\mathbb{Q}(v)$, the map κ gives an isomorphism between the affine q-Schur algebra $\widehat{S}_q(n,r)$ and the quotient of $U(\widehat{\mathfrak{gl}_n})$ by the kernel of its action on tensor space.*

Proof. By Lemma 3.4.6, it is enough to show that κ is a monomorphism. To do this, we consider an element in $\ker \kappa$ and then use the techniques of Lemma 3.3.4 to show that unless this element is zero, we have a counterexample in the case where v is specialized to 1, which contradicts Lemma 3.4.7. $\square$

4. Concluding results

4.1 Comparison of tensor space with q-tensor space.
We now show that the q-tensor space of §2.3 is isomorphic as an $\widehat{S}_q(n,r)$-module to tensor space $V^{\otimes r}$. In §4, we continue to use $\mathbb{Q}(v)$ as a base ring unless otherwise stated.

Lemma 4.1.1. *As a $U(\widehat{\mathfrak{gl}_n})$-module, tensor space $V^{\otimes r}$ is generated by any vector*

$$e_w = e_{w(1)} \otimes \cdots \otimes e_{w(r)}$$

for $w \in \widehat{W}$.

Proof. Using Lemma 3.4.3, it is enough to show that V_ω is generated as a $U(\widehat{\mathfrak{gl}_n})$-module by e_w. We introduce endomorphisms $\tau(T_{s_i}^{-1})$ of V_ω corresponding to the action of the elements $v^{-1}E_iF_i - 1$ of $U(\widehat{\mathfrak{gl}_n})$ on V_ω. It is easily checked that $\tau(T_{s_i}^{-1})$ is the inverse of $\tau(T_{s_i})$.

Let $e_{j_1} \otimes \cdots \otimes e_{j_r}$ be a basis element of V_ω. Let $1 \leq p(j,i) \leq r$ be the unique integer such that $j_{p(j,i)} \equiv i \mod n$. It is not hard to show that if $p(j,i) < p(j,i+1)$ then

$$\tau(T_{s_i})(e_{j_1} \otimes \cdots \otimes e_{j_r}) = v^* e_{s_i(j_1)} \otimes \cdots \otimes e_{s_i(j_r)},$$

for some integer $*$, and if $p(j,i) > p(j,i+1)$ (the only other possibility), we have

$$\tau(T_{s_i}^{-1})(e_{j_1} \otimes \cdots \otimes e_{j_r}) = v^* e_{s_i(j_1)} \otimes \cdots \otimes e_{s_i(j_r)}.$$

If $w = \rho^{\pm 1}$, we have

$$\tau(T_w)(e_{j_1} \otimes \cdots \otimes e_{j_r}) = v^* e_{w(j_1)} \otimes \cdots \otimes e_{w(j_r)}.$$

We can now use the techniques of Lemma 3.3.3 to deduce the claim. $\square$

Theorem 4.1.2. *As $\widehat{S}_q(n,r)$-modules, q-tensor space and tensor space are isomorphic. The isomorphism may be chosen to take $\phi^1_{\omega,\omega}$ to*

$$e_\omega = e_1 \otimes \cdots \otimes e_r.$$

Proof. By Lemma 4.1.1, we need only show that no element of q-tensor space annihilates the vector e_ω. Suppose

$$k = \sum_{\lambda \in \Lambda(n,r), d \in \mathcal{D}_\lambda} c_{\lambda,d} \phi^d_{\lambda,\omega}$$

annihilates e_ω. We may assume that only one λ occurs in the sum. In this case,

$$k = \phi^1_{\lambda,\omega} \cdot \left(\sum_{d \in \mathcal{D}_\lambda} c_{\lambda,d} \phi^d_{\omega,\omega} \right),$$

and if k acts as zero, so does

$$\sum_{w \in \widehat{W}_\lambda} \sum_{d \in \mathcal{D}_\lambda} c_{\lambda,d} \phi^{wd}_{\omega,\omega}.$$

(To see the latter assertion, premultiply by $\phi^1_{\omega,\lambda}$ and use Lemma 3.4.5.) This contradicts Lemma 3.3.4 unless all the $c_{\lambda,d}$ are zero. $\square$

Remark 4.1.3. The isomorphism between q-tensor space and tensor space is not a natural one as in the case of the ordinary q-Schur algebra or the case $q = 1$. To see this, compare the actions of $\phi^{s_r}_{\omega,\omega}$ on the generating vector: in the case of q-tensor space, this produces a combination of one other basis vector, but in the case of tensor space, e_1 occurs to the left of e_r in e_ω, so a combination of two basis vectors is produced.

4.2 Presentations for the affine Hecke algebra.

An important consequence of Theorem 4.1.2 is that it provides, in conjunction with Theorem 2.3.3, a right action of $\mathcal{H}$ on tensor space, and furthermore, the centralizing algebra of the action of $U(\widehat{\mathfrak{gl}_n})$ on tensor space is isomorphic to $\mathcal{H}$. This means that the endomorphisms y_i of Definition 3.2.4 can be regarded as elements of $\mathcal{H}$.

The results of this section consider this action further to show that our affine Hecke algebra is in fact isomorphic to the affine Hecke algebra in the sense of [**2**, 3.1 Definition].

Lemma 4.2.1. *Let*

$$e_{j_1} \otimes \cdots \otimes e_{j_r}$$

be a basis element of V_ω *satisfying* $1 \leq j_t \leq r$ *for all* t. *Suppose that* $j_i = r-1$ *and* $j_{i+1} = r$. *Then we have*

$$(e_{j_1} \otimes \cdots \otimes e_{j_r}).T_{s_i} y_i T_{s_i} = (e_{j_1} \otimes \cdots \otimes e_{j_r}).v^2 y_{i+1}.$$

Proof. By definition of the endomorphism y_{i+1} in Definition 3.2.4, the right hand side is easily seen to be

$$v^2 e_{j_1} \otimes \cdots \otimes e_{j_i} \otimes e_{-n+j_{i+1}} \otimes e_{j_{i+2}} \otimes \cdots \otimes e_{j_r}.$$

Because of the way $S_q(n,r)$ embeds in $\widehat{S}_q(n,r)$, the action of T_{s_i} on $V_n^{\otimes r}$ is the same as in the finite case. We therefore use the definition of this action in [**9**, §1.5] to show that

$$(e_{j_1} \otimes \cdots \otimes e_{j_r}).T_{s_i} y_i = v e_{j_1} \otimes \cdots \otimes e_{-n+j_{i+1}} \otimes e_{j_i} \otimes \cdots \otimes e_{j_r}.$$

It suffices to show that

$$(e_{j_1} \otimes \cdots \otimes e_{-n+j_{i+1}} \otimes e_{j_i} \otimes \cdots \otimes e_{j_r}).T_{s_i} = v(e_{j_1} \otimes \cdots \otimes e_{j_i} \otimes e_{-n+j_{i+1}} \otimes \cdots \otimes e_{j_r}).$$

Recall that $-n + j_{i+1} = -n + r$. There are no other indices with values between $-n+r-1$ and 0.

It is enough to establish

$$\begin{aligned}(e_{(n-r+1)+j_1} \otimes \cdots \otimes e_{1-r+j_{i+1}} \otimes e_{(n-r+1)+j_i} \otimes \cdots \otimes e_{(n-r+1)+j_r}).T_{s_i} \\ = v(e_{(n-r+1)+j_1} \otimes \cdots \otimes e_{(n-r+1)+j_i} \otimes e_{1-r+j_{i+1}} \otimes \cdots \otimes e_{(n-r+1)+j_r}),\end{aligned}$$

and then act

$$R^{-n+r-1} \in U(\widehat{\mathfrak{gl}_n})$$

on the left of both sides. Since all the indices lie between 1 and n and

$$n = (n-r+1) + j_i > 1 - r + j_{i+1} = 1,$$

the claim follows from the action of $\mathcal{H}$ on $V_n^{\otimes r}$. □

Corollary 4.2.2. *We have* $T_{s_i} y_i T_{s_i} = v^2 y_{i+1}$.

Proof. This follows from Lemma 4.1.1, Lemma 4.2.1 and the double centralizer property for $U(\widehat{\mathfrak{gl}_n})$ and $\mathcal{H}$ on tensor space. □

Lemma 4.2.3. *Suppose $1 \le i < r$, $1 \le j \le r$ and $j \ne i, i+1$. Let*

$$e_{k_1} \otimes \cdots \otimes e_{k_r}$$

be a basis element of V_ω satisfying $1 \le k_t \le r$ for all t. Suppose that $k_i = r-2$, $k_{i+1} = r-1$ and $k_j = r$. Then we have

$$(e_{k_1} \otimes \cdots \otimes e_{k_r}).y_j T_{s_i} = (e_{k_1} \otimes \cdots \otimes e_{k_r}).T_{s_i} y_j.$$

Proof. This is proved using an argument very similar to the that in proof of Lemma 4.2.1. □

Corollary 4.2.4. *We have $y_j T_{s_i} = T_{s_i} y_j$ if $j \ne i, i+1$.*

Proof. This uses Lemma 4.1.1, Lemma 4.2.3 and the double centralizer property. □

We can now establish an isomorphism between $\mathcal{H}$ and the affine Hecke algebra of [**2**, §3].

Theorem 4.2.5. *The algebra $\mathcal{H}(\widehat{W})$ is isomorphic to the algebra given by generators*

$$\{\sigma_i^{\pm 1} : 1 \le i < r\} \cup \{y_i^{\pm 1} : 1 \le i \le r\}$$

and defining relations

$$\begin{aligned}
\sigma_i \sigma_i^{-1} &= \sigma_i^{-1} \sigma_i = 1, && (1)\\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, && (2)\\
\sigma_i \sigma_j &= \sigma_j \sigma_i \text{ if } |i-j| > 1, && (3)\\
(\sigma_i + 1)(\sigma_i - v^2) &= 0, && (4)\\
y_j y_j^{-1} &= y_j^{-1} y_j = 1, && (5)\\
y_j y_k &= y_k y_j, && (6)\\
y_j \sigma_i &= \sigma_i y_j \text{ if } j \ne i, i+1, && (7)\\
\sigma_i y_i \sigma_i &= v^2 y_{i+1}. && (8)
\end{aligned}$$

The isomorphism may be chosen to identify σ_i with T_{s_i}.

Proof. We identify σ_i with T_{s_i}. Relations (1)–(4) are known from Definition 1.2.1. Relations (5) and (6) are obvious from considering the faithful action of $\mathcal{H}$ on tensor space. Relations (7) and (8) follow from corollaries 4.2.4 and 4.2.2 respectively.

These relations are sufficient since [**2**, 3.1 Lemma] shows that a basis for the algebra given by these generators and relations has a basis

$$y_1^{c_1} \cdots y_r^{c_r} T_w$$

as $c_i \in \mathbb{Z}$ and T_w ranges over a basis for $\mathcal{H}(\mathcal{S}_r)$.

Since the algebra in the statement is clearly isomorphic to its own opposite algebra, there is also a basis of the form

$$T_w y_1^{c_1} \cdots y_r^{c_r}.$$

It is not hard to verify that the map

$$T_w y_1^{c_1} \cdots y_r^{c_r} \mapsto e_\omega . T_w y_1^{c_1} \cdots y_r^{c_r}$$

is injective, from which we deduce that the algebra of [**2**] acts faithfully on tensor space under the identifications given. It follows that the relations given in the statement are sufficient. $\square$

Remark 4.2.6. Using Theorem 4.2.5 we can give a third construction of the affine q-Schur algebra more compatible with the results of [**2**]. This is formally the same as Definition 2.1.4, but uses the version of the affine Hecke algebra given in [**2**].

The presentation for $\mathcal{H}$ just given is more convenient when dealing with tensor space, in contrast to Proposition 1.2.3 which is adapted for q-tensor space. Using the basis $T_w \Pi y_i^{c_i}$ of $\mathcal{H}$ which was mentioned in the proof of the theorem, one can explicitly calculate the action of the generators T_{s_i} on tensor space. We do not do this here, because it is much more complicated than in the case of type finite A. In type finite A, the action of T_{s_i} on a basis element of $V_n^{\otimes r}$ produces a linear combination of at most two basis elements, but in the case of the action of T_{s_i} on $V^{\otimes r}$ in type affine A, it is possible for arbitrarily many basis elements to be involved. This happens because we are using a basis of $\mathcal{H}$ whose structure constants are much more complicated than those of the basis given in Proposition 1.2.3. (This complication arises from relation (8) in the statement of the theorem.)

4.3 Some questions.

It seems from the results of §3 that the Hopf algebra $U(\widehat{\mathfrak{gl}_n})$ is a natural object. Further evidence for this is provided by the results of [**25**, §11], where it is proved that the affine q-Schur algebra introduced in this paper occurs naturally in a K-theoretic context. This generalizes the interpretation in [**1**] of the ordinary q-Schur algebra in terms of the geometry of the action of the general linear group (over a finite field) acting on certain pairs of flags.

Comparing the situation with the relationship between $U(\mathfrak{gl}_n)$ and the ordinary q-Schur algebra raises some more questions about $U(\widehat{\mathfrak{gl}_n})$, such as the following ones.

Question 4.3.1. *Can one describe a suitable $\mathcal{A}$-form of the algebra $U(\widehat{\mathfrak{gl}_n})$ which maps under α_r to the natural $\mathcal{A}$-form of $\widehat{S}_q(n,r)$?*

This is an important question because it would show that our results also hold at roots of unity.

Question 4.3.2. *To what extent do these results hold in the case where $n < r$?*

In this case, there is no weight ω, and one should not expect the Hecke algebra to act faithfully on tensor space, since it does not act faithfully in the case of the ordinary q-Schur algebra. In the latter case, it can be shown that the centralizing algebra of the q-Schur algebra is a quotient of the Hecke algebra. Something similar may hold in the affine case.

A main obstacle to proving this lies in the difficulty of embedding $\widehat{S}_q(n,r)$ in $\widehat{S}_q(n+1,r)$. This is in contrast to the finite case, where there are obvious embeddings. The lack of an obvious determinant map also means that it is not clear how to realize $\widehat{S}_q(n,r)$ as a quotient of $\widehat{S}_q(n,r+n)$.

Question 4.3.3. *Is there a basis for $U(\widehat{\mathfrak{gl}_n})$ similar to Lusztig's canonical basis for the modified quantized enveloping algebras $\dot{U}$ which is compatible with the Kazhdan–Lusztig basis of §2.4 via the maps α_r?*

Lusztig's algebras $\dot{U}$ are defined in [**23**, §23]. They are equipped with canonical bases $\dot{B}$, generalizing the canonical bases for the algebras U^+ (which are the subalgebras of quantized enveloping algebras generated by the elements E_i). In the case of the ordinary q-Schur algebra, these bases have been shown in [**10**] to be compatible with the Kazhdan–Lusztig bases for the q-Schur algebra, in the sense that the basis elements $\dot{B}$ map either to zero or to Kazhdan–Lusztig basis elements. It seems reasonable to hope therefore that an analogous result exists for the affine case.

Question 4.3.4. *Do the structure constants for the affine q-Schur algebra with respect to the Kazhdan–Lusztig basis lie in $\mathbb{N}[v,v^{-1}]$?*

This property is true for the ordinary q-Schur algebra, as was proved in [**14**]. If the more general result is true, it should have an interpretation in terms of intersection cohomology.

Acknowledgements

The author thanks J. Du and R.J. Marsh for helpful conversations.

References

[**1**] A.A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of GL_n*, Duke. Math. J. **61** (1990), 655–677.

[**2**] V. Chari and A. Pressley, *Quantum affine algebras and affine Hecke algebras*, Pac. J. Math. **174** (1996), 295–326.

[**3**] I.V. Cherednik, *A unification of Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras*, Invent. Math. **106** (1991), 411–431.

[**4**] R. Dipper and G.D. James, *Representations of Hecke algebras and general linear groups*, Proc. L.M.S. **52** (1986), 20–52.

[**5**] R. Dipper and G.D. James, *The q-Schur algebra*, Proc. L.M.S. **59** (1989), 23–50.

[**6**] R. Dipper and G.D. James, *q-tensor space and q-Weyl modules*, Trans. A.M.S. **327** (1991), 251–282.

[**7**] R. Dipper, G.D. James and A. Mathas, *The (Q,q)-Schur algebra*, Proc. L.M.S. **77** (1998), 327–361.

[**8**] J. Du, *Kazhdan-Lusztig bases and isomorphism theorems for q-Schur algebras*, Contemp. Math. **139** (1992), 121–140.

[**9**] J. Du, *A Note on Quantized Weyl Reciprocity at Roots of Unity*, Alg. Colloq. **4** (1995), 363–372.

[**10**] J. Du, *Global Bases for Quantum Linear Groups*, J. Pure Appl. Alg. (to appear).

[**11**] J. Du and L. Scott, *The q-Schur^2 algebra* (preprint).

[**12**] R.M. Green, *A matrix setting for the q-Schur algebra*, Jour. L.M.S. (in press).

[**13**] R.M. Green, *q-Schur Algebras as Quotients of Quantized Enveloping Algebras*, J. Alg. **185** (1996), 660–687.

[**14**] R.M. Green, *Positivity Properties for q-Schur Algebras*, Proc. Cam. Phil. Soc. **122** (1997), 401–414.

[**15**] R.M. Green, *Hyperoctahedral Schur algebras*, J. Alg. **192** (1997), 418–438.

[**16**] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.

[**17**] J.C. Jantzen, *Lectures on Quantum Groups*, Amer. Math. Soc., Providence, R.I., 1995.

[**18**] M. Jimbo, *A q-analogue of $\mathcal{U}(gl(N+1))$, Hecke algebras and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), 247–252.

[**19**] V.F.R. Jones, *A quotient of the affine Hecke algebra in the Brauer algebra*, L'Enseignement Math. **40** (1994), 313–344.

[**20**] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[**21**] D. Kazhdan and G. Lusztig, *Proof of the Deligne–Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), 153–215.

[**22**] G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc. **277** (1983), 623–653.

[**23**] G. Lusztig, *Introduction to Quantum Groups* (1993), Birkhäuser, Basel.

[**24**] H. Rui, *On endomorphism algebras arising from multiparameter Hecke algebras*, J. Alg. **195** (1997), 308–319.

[**25**] M. Varagnolo and E. Vasserot, *Canonical bases and Lusztig conjecture for quantized $SL(N)$ at roots of unity* (q-alg preprint 9803023).