MATH 8174: Assignment 10

1. Let k be a field and let $\mathfrak{u}_n(k)$ denote the Lie algebra of strictly upper triangular $n \times n$ matrices over k, equipped with the usual Lie bracket. Show that $\mathfrak{u}_n(k)$ is a nilpotent subalgebra of $\mathfrak{gl}_n(k)$. What is the nilpotency class of $\mathfrak{u}_3(\mathbb{C})$?

2. Let L be a Lie algebra. Define $Z_0(L) = 0$ and, for $i \geq 0$, define $Z_{i+1}(L)$ by the condition
$$Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L)),$$
where Z denotes the center of a Lie algebra.
Show that the elements of the chain $Z_0(L) \leq Z_1(L) \leq \cdots$ are ideals of L. This chain is called the *upper central series* (or *ascending central series*) of L. Show that L is nilpotent of class c if and only if c is the smallest integer for which $Z_c = L$.

3. Let V be a finite dimensional vector space. Show that the Lie algebra $\mathrm{End}_k(V)$ contains a nilpotent subalgebra N (in the Lie algebra sense) such that (a) $\dim N = \dim V$ and (b) N has no nonzero nilpotent elements (in the associative sense).

4. Let L be a Lie algebra of dimension n over k. Show that the adjoint representation of L, together with a choice of basis for L, yields a copy of $L/Z(L)$ inside $\mathfrak{gl}_n(k)$. If, in addition, L is a nilpotent Lie algebra, show that the aforementioned copy is conjugate to a subalgebra of $\mathfrak{u}_n(k)$.

5. Let J and K be ideals of a Lie algebra L. Show that we have
$$(J + K)^{2n} \leq J^n + K^n.$$
Deduce that the set of nilpotent ideals of a finite dimensional Lie algebra has a unique maximal element. (This maximal element is sometimes called the *nilradical* of L.)