MATH 8174: Assignment 3

1. Let G be a finite p-group and let V be a representation of G over a field of characteristic p. Let v be a nonzero element of V. By considering the action of G on the finite additive abelian p-group inside V generated by the elements $g(v)$ for $g \in G$, show that there is a nonzero vector in V that is fixed by all elements $g \in G$. Deduce that the only irreducible representation of a finite p-group in characteristic p is the trivial representation.

   From now on, all questions are about representations of finite groups over the field of complex numbers, except where otherwise stated.

2. Write down the conjugacy classes of the alternating group A_5. What is the character of the 3-dimensional irreducible representation of A_5 acting by rotations on the regular dodecahedron?
3. Let Z be the center of the group G, and let V be an irreducible representation of G. Show that each element of Z acts as a scalar multiple of the identity. Deduce that the character value of each $z \in Z$ is a complex number whose absolute value is equal to the dimension of V.
4. Suppose that V and W are two representations of G, and denote by $\mathrm{Hom}_{\mathbb{C}}(V, W)$ the $\mathbb{C}$-vector space of all $\mathbb{C}$-linear maps from V to W. Show that $\mathrm{Hom}_{\mathbb{C}}(V, W)$ may be made into a representation of G as follows. If $g \in G$, $v \in V$ and $f : V \to W$, define $g(f) : V \to W$ by $(g(f))(v) = g(f(g^{-1}(v)))$. Show that there is an equivalence of representations $V^* \otimes W \xrightarrow{\theta} \mathrm{Hom}_{\mathbb{C}}(V, W)$ given as follows. If $\alpha : V \to \mathbb{C}$, $v \in V$ and $w \in W$, then $\theta(\alpha \otimes w)(v) = \alpha(v).w$.
5. Let G be the symmetric group S_4. Find a complete set of inequivalent irreducible representations of G. Show that your list is complete by adding up the squares of the dimensions. Write down the character table of G.