MATH 8174: Assignment 2

1. (Modular Law) Suppose that A, B and C are submodules of a module M over a ring R, and suppose that $A \leq B$. Show that

$$(B \cap C) + A = B \cap (C + A).$$

2. Let G be the Klein four group, given by the presentation

$$\langle x, y : x^2 = y^2 = 1,\ xy = yx \rangle.$$

Let K be an infinite field of characteristic 2. Show that the representations

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad y \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

are indecomposable and non-isomorphic for different values of a. Deduce that there are infinitely many isomorphism classes of indecomposable KG-modules.

3. Use Jordan canonical form to classify the isomorphism classes of indecomposable representations of a cyclic group of order p over an algebraically closed field of characteristic p. Show that there are only finitely many such isomorphism classes.

4. Let R be a ring. A *short exact sequence* of R-modules is a sequence of modules and homomorphisms

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

where (i) α is injective, (ii) β is surjective, and (iii) $\ker(\beta) = \mathrm{Im}(\alpha)$. In other words, A is isomorphic to a submodule of B with quotient isomorphic to C. Such a short exact sequence is said to *split* if there is a homomorphism $\gamma : C \to B$ with $\beta \circ \gamma$ equal to the identity map on C. If G is a finite group, show that every short exact sequence of finite dimensional $\mathbb{C}G$-modules splits. Give an example of a short exact sequence of KG-modules (for some field K and some group G) that does not split.

5. Calculate the character of the 3-dimensional representation of the symmetric group S_4 as rotations of a cube. (You need to list the conjugacy classes of S_4 first.)