
Put your name on each answer sheet. Answer all three questions.
Justify all your answers in full.
Formula sheets, calculators, notes and books are not permitted.

1. Let $R = \mathbb{Z}$ and consider $M = \mathbb{Q}/\mathbb{Z}$ as an $R$-module in the usual way. Show that the tensor algebra $T(\mathbb{Q}/\mathbb{Z})$ is isomorphic as a $\mathbb{Z}$-module to $\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$. Under these identifications, show that the subset of $T(\mathbb{Q}/\mathbb{Z})$ given by

$$I = \{(5z, q + \mathbb{Z}) : z \in \mathbb{Z}, q \in \mathbb{Q}\}$$

is a graded ideal of $T(\mathbb{Q}/\mathbb{Z})$. Describe the quotient ring $T(\mathbb{Q}/\mathbb{Z})/I$.

2. Let $n > 1$ and let $B \in M_n(\mathbb{Q})$ be the matrix all of whose entries are equal to 1. You may assume without proof that the vectors

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
\vdots \\
1
\end{pmatrix}, \quad
\begin{pmatrix}
-1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
1 \\
0 \\
\vdots \\
-1
\end{pmatrix}, \quad \ldots, \quad
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
1
\end{pmatrix}
\]

form a basis of $\mathbb{Q}^n$ consisting of eigenvectors for $B$. Find the Jordan canonical form for $B$, and use it to deduce the rational canonical form for $B$ and the minimal polynomial of $B$.

3. Let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{2}, i)$ be subfields of $\mathbb{C}$. Find the degree of $K$ over $F$, and write down a basis for $K$ as an $F$-vector space. Prove that the polynomial $x^2 - 2$ is irreducible over the field $\mathbb{Q}(i)$. 