Math 6320 Homework #9 Solution

11, 1. *Let* $\{A_n\}$ *be a countable collection of measurable sets. Then*

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} A_k\right).$$

Let $B_n = \cup_{k=1}^n A_k$. Then each B_n is measurable, and $B_n \subset B_{n+1}$ for each $n \in \mathbb{N}$, and obviously $\cup_{n=1}^\infty B_n = \cup_{k=1}^\infty A_k$.

Let $C_n = B_n \backslash B_{n-1}$ for $n > 1$, with $C_1 = B_1$. Then if $m \neq n$, then $C_n \cap C_m = \emptyset$, since (assuming $n > m$)

$$C_n \cap C_m = B_n \cap \widetilde{B_{n-1}} \cap B_m \cap \widetilde{B_{m-1}},$$

and $B_m \subset B_{n-1}$ implies $\widetilde{B_{n-1}} \cap B_m = \emptyset$.

Furthermore for any n we have $\cup_{k=1}^n C_k = \cup_{k=1}^n B_k$, since inductively

$$\begin{aligned} C_{n+1} \cup \bigcup_{k=1}^{n} C_k = \left(\widetilde{B_n} \cap B_{n+1}\right) \cup \bigcup_{k=1}^{n} B_k &= \left(\widetilde{B_n} \cup \bigcup_{k=1}^{n} B_k\right) \cap \left(B_{n+1} \cup \bigcup_{k=1}^{n} B_k\right) \\ &= X \cap \left(\bigcup_{k=1}^{n+1} B_k\right) = \bigcup_{k=1}^{n+1} B_k. \end{aligned}$$

Hence in particular $\cup_{k=1}^\infty C_k = \cup_{k=1}^\infty B_k$.

So all C_n are measurable, and all disjoint, and we can use countable additivity. Thus we have

$$\begin{aligned} \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \mu\left(\bigcup_{k=1}^{\infty} C_k\right) &= \sum_{k=1}^{\infty} \mu C_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu C_k = \lim_{n\to\infty} \mu B_1 + \sum_{k=2}^{n} \mu B_k - \mu B_{k-1} \\ &= \lim_{n\to\infty} \mu B_n = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} A_k\right). \end{aligned}$$

11, 6b. *Show that each measure μ is the sum $\mu_1 + \mu_2$ of a semifinite measure μ_1 and a measure μ_2 that assumes only the values 0 and ∞.*

By definition, a measure μ_1 is semifinite if and only if every measurable set of infinite measure contains sets of arbitrarily large finite measure. Another way of thinking of this is that μ_1 is semifinite iff for any measurable set A, we have $\mu_1 A$ equal to the supremum of measures of subsets with finite measure.

So for any $E \in \mathcal{B}$, define $\mu_1 \emptyset = 0$ and

$$\mu_1 E = \sup_{A \subset E, \mu A < \infty} \mu A.$$

Clearly $\mu_1 E \leq \mu E$ for every E, and if $\mu E < \infty$ then $\mu_1 E = \mu E$.

By construction, μ_1 is semifinite, and we just have to check that μ_1 satisfies the measure axioms. By construction $\mu_1 \emptyset = 0$. So we just need to check countable additivity.

Let E_i be a countable collection of disjoint measurable sets and let $E = \cup_{i=1}^{\infty} E_i$. We want to prove that

$$\mu_1(E) = \sum_{i=1}^{\infty} \mu(E_i).$$

For any set $A \subset E$ with $\mu A < \infty$, we have

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A \cap E_i) \leq \sum_{i=1}^{\infty} \mu_1(E_i).$$

Since this is true for every $A \subset E$ with finite measure, we have

$$\mu_1(E) \leq \sum_{i=1}^{\infty} \mu_1(E_i).$$

On the other hand, if $\mu_1(E) < \infty$ then $\mu_1(E_i) < \infty$ for every i, and for any $\varepsilon > 0$ we can choose $A_i \subset E_i$ such that $\mu_1(E_i) - \frac{\varepsilon}{2^i} < \mu(A_i) < \infty$ (by definition of μ_1). Thus all the A_i are also disjoint, and

$$\mu_1(E) \geq \sum_{i=1}^{\infty} \mu(A_i) \geq \sum_{i=1}^{\infty} \mu_1(E_i) - \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \sum_{i=1}^{\infty} \mu_1(E_i) - \varepsilon.$$

This is true for any $\varepsilon > 0$, so that

$$\mu_1(E) \geq \sum_{i=1}^{\infty} \mu(E_i).$$

Thus μ_1 is countably additive.

Now if we want for μ_2 to also be a measure such that $\mu = \mu_1 + \mu_2$, then we must have $\mu_2 \emptyset = 0$. If $\mu E < \infty$ then $\mu_1 E = \mu E$, so that we must have $\mu_2 E = 0$. If E is σ-finite, then $\mu E = \infty$ and $\mu_1 E = \infty$ from above. Thus if E is σ-finite, we also must have $\mu_2 E = 0$. If $\mu_1 E < \mu E$, then $\mu E = \infty$, so that we must have $\mu_2 E = \infty$. In that case E cannot be σ-finite. The only cases that aren't determined are when E is not σ-finite but $\mu_1 E = \infty$. We may as well define $\mu_2 E = \infty$ in this case.

Take any collection of disjoint measurable sets E_i and let $E = \cup_{i=1}^{\infty} E_i$. If every E_i is σ-finite then also E is σ-finite; we just write $E_i = \cup_{j=1}^{\infty} E_{ij}$ where $\mu E_{ij} < \infty$. Then letting $F_n = \cup_{i,j=1}^{n} E_{ij}$, we see E is σ-finite. Similarly if E is σ-finite, then each E_i is σ-finite. So we get countable additivity in this case. If E is not σ-finite, then at least one E_i is not σ-finite, so both sides are infinite and we again have countable additivity.

11, 7. *Prove Proposition 4: If $(X, \mathcal{B}, \mu)$ is a measure space, then we can find a complete measure space $(X, \mathcal{B}_0, \mu_0)$ such that*

- *i* $\mathcal{B} \subset \mathcal{B}_0$.
- *ii* $E \in \mathcal{B} \Rightarrow \mu E = \mu_0 E$.
- *iii* $E \in \mathcal{B}_0 \Leftrightarrow E = A \cup B$ *where* $B \in \mathcal{B}$ *and* $A \subset C$, $C \in \mathcal{B}$, $\mu C = 0$.

[First show that the family $\mathcal{B}_0$ defined by (iii) is a σ-algebra. If $E \in \mathcal{B}_0$, show that μA is the same for all sets $A \in \mathcal{B}$ such that $E = A \cup B$ with B a subset of a set of measure zero. Use this fact to define μ_0 and show μ_0 is a measure.]

Following the hint, we define $\mathcal{B}_0$ by (iii). If $E \in \mathcal{B}_0$, we want $\tilde{E} \in \mathcal{B}_0$; writing $E = A \cup B$ with $B \in \mathcal{B}$ and $A \subset C$ with $\mu C = 0$, we have $\tilde{E} = \tilde{A} \cap \tilde{B}$. Now $A \subset C$ implies $\tilde{C} \subset \tilde{A}$, and thus $\tilde{B} \cap \tilde{A} \supset \tilde{B} \cap \tilde{C}$. So we can write

$$\begin{aligned}\tilde{B} \cap \tilde{A} &= (\tilde{B} \cap \tilde{C}) \cup \big((\tilde{B} \cap \tilde{A}) \backslash (\tilde{B} \cap \tilde{C})\big) \\ &= (\tilde{B} \cap \tilde{C}) \cup \Big((\tilde{B} \cap \tilde{A}) \cap (B \cup C)\Big) \\ &= (\tilde{B} \cap \tilde{C}) \cup \Big((B \cap \tilde{B} \cap \tilde{A}) \cup (C \cap \tilde{B} \cap \tilde{A})\Big) \\ &= (\tilde{B} \cap \tilde{C}) \cup (C \cap \tilde{B} \cap \tilde{A})\end{aligned}$$

Now $\tilde{B} \cap \tilde{C} \in \mathcal{B}$, and $C \cap \tilde{B} \cap \tilde{A} \subset C$. So $\tilde{E} \in \mathcal{B}_0$.

Next we have to check countable unions: if $E_i = B_i \cup A_i$ where $A_i \subset C_i$ with $\mu C_i = 0$, then $C = \cup_{i=1}^{\infty} C_i$ also has measure zero, and $A = \cup_{i=1}^{\infty} A_i$ is a subset of C. Furthermore $B = \cup_{i=1}^{\infty} B_i \in \mathcal{B}$. So $\cup_{i=1}^{\infty} E_i = B \cup A \in \mathcal{B}_0$. Thus $\mathcal{B}_0$ is a σ-algebra containing $\mathcal{B}$.

We want to define $\mu_0(B \cup A) = \mu B$ whenever $A \subset C$ with $C, B \in \mathcal{B}$ and $\mu C = 0$. We just need to check that this is consistent: if $B \cup A = D \cup E$ where $B, D \in \mathcal{B}$, and $A \subset C$ and $E \subset F$ with $\mu C = \mu F = 0$, then is $\mu B = \mu D$? To see that this is so, note that $B \subset B \cup A = D \cup E \subset D \cup F$, so that $\mu B \leq \mu D + \mu F = \mu D$; similarly the other way around.

Finally we need to check that μ_0 defined this way is a measure. Let $E_i = B_i \cup A_i$ with E_i all disjoint and B_i, A_i as before. Then all B_i are disjoint, and we know that $\mu_0 E_i = \mu B_i$. Also $E = \cup_{i=1}^{\infty} = (\cup_{i=1}^{\infty} B_i) \cup (\cup_{i=1}^{\infty} A_i)$ and

$$\mu_0 E = \mu(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu B_i = \sum_{i=1}^{\infty} \mu_0 E_i.$$

11, 13a. *A sequence $\langle f_n \rangle$ of measurable real-valued functions is said to* converge in measure *to a function f if given $\epsilon > 0$, there is an integer N and a measurable set E with $\mu E < \epsilon$ such that*

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$ and all $x \notin E$. Show that, if f_n converges to f in measure, then there is a subsequence f_{n_k} that converges to f almost everywhere.

For each k, choose n_k and E_k such that $n_k > n_{k-1}$, $\mu E_k < \frac{1}{2^k}$, and $|f_{n_k}(x) - f(x)| < \frac{1}{2^k}$ for all $x \notin E_k$. Set $F_n = \cup_{k=n}^{\infty} E_k$ and $F = \cap_{n=1}^{\infty} F_n$.

Then $\mu F_n \leq \frac{1}{2^{n-1}}$ so that $\mu F = 0$. If $x \notin F$ then $x \notin F_m$ for some m, and hence $x \notin E_k$ for all $k \geq m$. Thus $|f_{n_k}(x) - f(x)| < \frac{1}{2^k}$ for all $k \geq m$; in particular $\lim_{k\to\infty} f_{n_k}(x) = f(x)$.

So f_{n_k} converges to f almost everywhere.

11, 13b. *Suppose that $\langle f_n \rangle$ is a sequence of measurable functions each of which vanishes outside a fixed measurable set A with $\mu A < \infty$. Suppose that $f_n(x) \to f(x)$ for almost all x. Then $\langle f_n \rangle$ converges to f in measure.*

We may as well assume that $f_n \to f$ for all $x \in A$, by taking a smaller set A with the same measure. For $m \in \mathbb{N}$ and $n \in \mathbb{N}$, let

$$A_{mn} = \{x \in A \,|\, |f_n(x) - f(x)| < \tfrac{1}{m}\}.$$

For any x and any fixed m, we know there is an N such that $n \geq N$ implies $x \in A_{mn}$. In other words, if $B_{mN} = \cap_{n=N}^{\infty} A_{mn}$, then for any m we have $x \in \cup_{N=1}^{\infty} B_{mN}$. Of course we also have $B_{mN} \subset B_{m,N+1}$ for every m and N. Thus for every $x \in A$ we have

$$x \in \cap_{m=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n=N}^{\infty} A_{mn}.$$

Yet another way of stating this is that for any m we have

$$A = \cup_{N=1}^{\infty} B_{mN},$$

and thus $\mu A = \lim_{N\to\infty} \mu B_{mN}$. Thus for any $\varepsilon > 0$ we can choose $m > \frac{1}{\varepsilon}$, and there is also an N such that $\mu(B_{mN}) > \mu(A) - \varepsilon$, or in other words $\mu(A \backslash B_{mN}) < \varepsilon$. This set $E = A \backslash B_{mN}$ is our desired set; its measure is smaller than ε, and on its complement either $f_n(x) = f(x) = 0$ or $|f_n(x) - f(x)| < \frac{1}{m} < \varepsilon$ for every $n \geq N$.