Math 6320 Homework #8 Solution

10, 26. *Let $\langle T_n \rangle$ be a sequence of continuous linear operators on a Banach space X to a normed vector space Y, and suppose that for each $x \in X$ the sequence $\langle T_n x \rangle$ converges to a value Tx. Then T is a bounded linear operator.*

Since for every x, we have $T_n x \to Tx$, we know $\{T_n x \,|\, n \in \mathbb{N}\}$ is bounded for each x. Hence T_n satisfies pointwise boundedness, so it satisfies all the hypotheses of the uniform boundedness principle. Hence there is a uniform bound M for all T_n, and so $\|T\| \le M$.

Furthermore $|T_n(\alpha x + \beta y) = \alpha T_n(x) + \beta T_n(y)$ for every n as well as every $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$, and thus by taking pointwise limits we have $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$. So the limit T is actually linear.

We conclude T is a bounded linear operator.

10, 28. *Let S be a linear subspace of $C[0,1]$ which is closed as a subspace of $L^2[0,1]$.*

(a) *Show that S is a closed subspace of $C[0,1]$.*

Let $\langle f_n \rangle$ be a sup-Cauchy sequence in S. Then $\langle f_n \rangle$ is also an L^2-Cauchy sequence in S, since

$$\|f_n - f_m\|_{L^2} = \sqrt{\int_0^1 |f_n(x) - f_m(x)|^2 \, dx} \le \sup_{x \in [0,1]} |f_n(x) - f_m(x)| = \|f_n - f_m\|_C.$$

Since S is closed in $L^2[0,1]$, it is complete in the L^2 norm, and thus f_n converges to some $f \in L^2$, with $f \in S$. But also $C[0,1]$ is complete, and so $\langle f_n \rangle$ converges to some continuous g (which we don't yet know is in S). However if $f_n \to g$ in the supremum norm, then also $f_n \to g$ in L^2.

But if $f_n \to f$ in L^2 and $f_n \to g$ in L^2, then $f = g$ almost everywhere. So in fact $g \in S$.

(b) *Show that there is a constant M such that for all $f \in S$ we have $\|f\|_2 \le \|f\|_\infty$ and $\|f\|_\infty \le M\|f\|_2$.*

S is a Banach space in either topology, and $\|f\|_{L^2} \le \|f\|_C$ for all f in S. Hence by Proposition 11, we must have a constant M such that

$$\|f\|_C \le \|f\|_{L^2}$$

for all f in S.

(c) *Show that for each $y \in [0,1]$ there is a function k_y in L^2 such that for each $f \in S$ we have $f(y) = \int k_y(x) f(x)\,dx$.*

For each y let $F_y \colon S \to \mathbb{R}$ be the linear functional $F_y(f) = f(y)$. Then F_y is obviously bounded as a functional on S_C. It is also bounded as a functional on S_{L^2}, since by part (b) there is an M such that for each f in S,

$$|F_y(f)| = f(y) \leq \|f\|_C \leq M\|f\|_{L^2}.$$

Now F_y is a bounded linear functional on S, a subspace of $L^2[0,1]$. By the Hahn-Banach theorem, we can extend F_y to be defined on *all* of $L^2[0,1]$ in such a way that it agrees with F_y on S. Now by the Riesz representation theorem (Theorem 6.13), every functional on L^2 can be written as

$$F_y(f) = \int_0^1 k_y(x) f(x)\,dx.$$

Of course we won't have $F_y(f) = f(y)$ for all $f \in L^2[0,1]$, but we will have $F_y(f) = f(y)$ for all $f \in S$, which is all we need.

10, 37. *Prove that every locally compact Hausdorff vector space X is finite-dimensional. [Hint: Let V be a neighborhood of θ with $\overline{V}$ compact and $\alpha V \subset V$ for each α, $|\alpha| < 1$. If we cover $\overline{V}$ by a finite number of translates $x_1 + \frac{1}{3}V, \ldots, x_n + \frac{1}{3}V$, then $x_1, \ldots, x_n$ form a basis for X.]*

We know by Proposition 14 that there is a base at θ consisting of sets satisfying $\alpha V \subset V$ for $|\alpha| \leq 1$. Since we can find at least open set U containing θ whose closure is compact, we can find a basis element inside U whose closure will also be compact. Hence we get the desired V.

Now for each x, the translate $x + \frac{1}{3}V$ is also open, and the union over all $x \in \overline{V}$ covers $\overline{V}$. So there is a finite set $\{x_1, \ldots, x_n\}$ such that

$$V \subset \overline{V} \subset (x_1 + \tfrac{1}{3}V) \cup \cdots \cup (x_n + \tfrac{1}{3}V).$$

Now we want to prove every $y \in V$ can be expressed as a linear combination of the $\{x_i\}$. Take any $y_0 \in V$. Then $y_0 \in x_{k_1} + \frac{1}{3}V$ for some k_1, and hence

$$y_0 = x_{k_0} + \tfrac{1}{3}y_1$$

for some $y_1 \in V$. Now we can repeat this to get

$$y_1 = x_{k_1} + \tfrac{1}{3} y_2,$$

which implies

$$y_0 = x_{k_0} + \tfrac{1}{3} x_{k_1} + \tfrac{1}{9} y_2,$$

and so on, which implies that

$$y_0 = \sum_{j=0}^{n-1} \tfrac{1}{3^j} x_{k_j} + \tfrac{1}{3^n} y_n$$

for every n.

We just need to prove that $\frac{1}{3^n} y_n$ converges to θ. This comes from the assumption that the space is Hausdorff: recall from Proposition 14 that X is Hausdorff if and only if for any basis element V, the intersection $\bigcap_{\alpha>0} \alpha V = \{\theta\}$; hence in particular $\bigcap_{n=1}^{\infty} \frac{1}{3^n} V = \{\theta\}$. Choose any open $U \ni \theta$ such that $U \subset V$, and find a number $\alpha > 0$ such that $\theta \in \alpha V \subset U$, and find an integer N such that $\frac{1}{3^N} < \alpha$. Then $n > N$ implies $\frac{1}{3^n} y_n \in \frac{1}{3^n} V \subset \frac{1}{3^N} V \subset U$, so indeed $\frac{1}{3^n} y_n$ converges to 0.

We have thus shown that every $y_0 \in V$ can be expressed as a linear finite combination of the $\{x_i\}$. Since (once again by Proposition 14) $\bigcup_{\alpha>0} \alpha V = X$, we see that every vector in X can be expressed in terms of the $\{x_i\}$, so $\{x_i\}$ spans. Hence a subset of it forms a basis, and X is finite-dimensional.

10, 38a. *Show that if $x_n \to x$ weakly, then $\langle \|x_n\| \rangle$ is bounded.*

This comes from the uniform boundedness principle. We know that for each $f \in X*$ that $\lim_{n\to\infty} f(x_n) = f(x)$. In particular, for each f there is a number M_f such that $|f(x_n)| \le M_f$. Now thinking in terms of elements of X^{**}, this says that the family $\varphi(x_n) \in X^{**}$ is pointwise bounded. Hence there is a constant M such that $\|\varphi(x_n)\|_{X^{**}} \le M$ for all n. Since $\|\varphi(x_n)\|_{X^{**}} = \|x_n\|_X$, this shows $\|x_n\|$ is bounded.

10, 41. *Let S be the linear subspace of $C[0,1]$ given in Problem 28.*

(a) *Show that if $f_n \to f$ weakly in L^2, then $f_n(y) \to f(y)$ for each $y \in [0,1]$.*

By Problem 28b there is a constant M such that $\|f\|_C \leq M\|f_n\|_{L^2}$ for each $f \in S$. In particular, since S is a subspace, we know $f_n - f \in S$, so $\|f_n - f\|_C \leq M\|f_n - f\|_{L^2}$. Hence f_n converges to f in the supremum norm, and in particular $f_n(y)$ converges to $f(y)$ for every y (not just almost everywhere).

(b) *If $f_n \to f$ weakly in L^2, then $\|f_n\|_\infty$ is bounded, and hence $f_n \to f$ strongly in L^2 by the Lebesgue convergence theorem.*

By Problem 38a, we know $f_n \to f$ weakly implies f_n is bounded in L^2. Hence f_n is bounded also in the supremum norm (again using Problem 28b). The Lebesgue convergence theorem (Theorem 4.16) says that if $(f_n - f)^2 \to 0$ pointwise and $(f_n - f)^2$ is uniformly bounded by some integrable function (in particular, uniformly bounded by a constant), then

$$\lim_{n\to\infty} \int_0^1 (f_n - f)^2 = \int_0^1 0 = 0.$$

So $f_n \to f$ strongly, in the L^2 norm.

(c) *The space S is a locally compact subspace of L^2 and hence finite-dimensional.*

Since L^2 is a metric space, so is S, and hence a subset is compact if and only if it is sequentially compact. Since L^2 is reflexive, the weak topology is the same as the weak-* topology, and hence the intersection of S with the unit ball in L^2 is compact in the weak topology by Alaoglu's theorem.

So any bounded sequence in S has a weakly convergent subsequence, and hence a strongly convergent subsequence, and so the closed unit ball in S is sequentially compact (hence compact). So S is locally compact in the strong topology.

We want to use Problem 37; we just need to check S is Hausdorff. But this is true since any topological subspace of a Hausdorff space is Hausdorff. So S is locally compact and Hausdorff in the strong topology, and thus S is finite-dimensional.