Math 6320 Homework #7 Solution

10, 17. *Show that a linear functional f on a normed linear space is bounded iff its kernel is closed. [The kernel of f is $\{x : f(x) = 0\}$.]*

If f is bounded, then it is uniformly continuous by Proposition 10.2. Hence since $\{0\}$ is a closed set in $\mathbb{R}$, $f^{-1}\{0\}$ is closed.

Now suppose the kernel K of f is closed. Let x be any vector not in K, so that $f(x) \neq 0$; without loss of generality we may assume $f(x) = 1$. Then any $y \in X$ can be written as $y = \lambda x + k$ for some $\lambda \in \mathbb{R}$ and some $k \in K$; just choose $\lambda = f(y)$, and we know that $f(y - \lambda x) = f(y) - \lambda f(1) = f(y) - f(y) = 0$, and hence $y - \lambda x \in K$.

We want to prove f is bounded, i.e., that

$$\sup_{\lambda \neq 0, k \in K} \frac{|\lambda|}{|\lambda x + k|} < \infty.$$

Dividing top and bottom by $-\lambda$, and observing that $k \in K$ iff $-\frac{k}{\lambda} \in K$ since K is a subspace, we see that it's enough to prove

$$\sup_{k \in K} \frac{1}{\|k - x\|} < \infty,$$

i.e., that $\inf_{k \in K} \|k - x\| > 0$. This is true since K is closed; if this infimum were zero, there would be a sequence in K converging to x, which would imply $x \in K$.

10, 20. *Let l^∞ be the space of all bounded sequences. Use Proposition 5 to show that there is a linear functional F on l^∞ with the following properties:*

i. $\underline{\lim} \xi_n \leq F[\langle \xi_n \rangle] \leq \overline{\lim} \xi_n$.

ii. $F[\langle \xi_n + \eta_n \rangle] = F[\langle \xi_n \rangle] + F[\langle \eta_n \rangle]$.

iii. $F[\langle \alpha \xi_n \rangle] = \alpha F[\langle \xi_n \rangle]$.

iv. *If* $\eta_n = \xi_{n+1}$, *then* $F[\langle \eta_n \rangle] = F[\langle \xi_n \rangle]$.

Parts (ii) and (iii) are automatic as long as we show that F is linear, so we only have to worry about (i) and (iv). Furthermore, only half of part (i) is really needed: if we knew that

$$F[\langle \xi_n \rangle] \leq \limsup_{n \to \infty} \xi_n$$

for every bounded sequence $\langle \xi_n \rangle$, then (replacing ξ with $-\xi$) we'd also know that

$$-F[\langle \xi_n \rangle] \leq \limsup_{n\to\infty}(-\xi_n) = -\liminf_{n\to\infty} \xi_n,$$

so

$$F[\langle \xi_n \rangle] \geq \liminf_{n\to\infty} \xi_n.$$

So at this point it's clear what we have to do: the linear operators A_k given by $A_k[\langle \xi_1, \xi_2, \xi_3, \ldots \rangle] = \langle \xi_{k+1}, \xi_{k+2}, \xi_{k+3}, \ldots \rangle$ satisfy A_0 equals the identity, and $A_j A_k = A_{j+k} = A_k A_j$ for all $j, k \in \mathbb{N} \cup \{0\}$.

Let S be the set of all convergent real sequences. Since

$$\lim_{n\to\infty} \lambda\eta_n + \mu\xi_n = \lambda \lim_{n\to\infty} \eta_n + \mu \lim_{n\to\infty} \xi_n,$$

we know S is a subspace. Define $f\colon S \to \mathbb{R}$ by $f[\langle \xi_n \rangle] = \lim\limits_{n\to\infty} \xi_n$. Clearly f is linear, and since the limit of a convergent sequence does not change if finitely many terms are deleted, $f \circ A_k = f$ on S.

Now define $p\colon l^\infty \to \mathbb{R}$ by

$$p[\langle \xi_n \rangle] = \limsup_{n\to\infty} \xi_n;$$

this is well-defined since every bounded sequence has a finite limit superior. p is not linear, but it does satisfy the only axioms we need:

- $\limsup\limits_{n\to\infty} \alpha\xi_n = \alpha \limsup\limits_{n\to\infty} \xi_n$ if $\alpha \geq 0$,
- $\limsup\limits_{n\to\infty}[\eta_n + \xi_n] \leq \limsup\limits_{n\to\infty} \eta_n + \limsup\limits_{n\to\infty} \xi_n$ for any two bounded sequences, and
- $\limsup\limits_{n\to\infty} \eta_{n+k} = \limsup\limits_{n\to\infty} \eta_n$ for any $k \geq 0$.

(If we had taken instead $p(\langle \xi_n \rangle) = \sup\limits_{n\in\mathbb{N}} \xi_n$, we'd have $p(A_k\langle \xi_n \rangle) \leq p(\langle \xi_n \rangle)$, which is all we actually need.)

Finally we obviously have for any convergent sequence that

$$\lim_{n\to\infty} \xi_n = \limsup_{n\to\infty} \xi_n,$$

so that $f = p$ on S.

So the Hahn-Banach theorem says there is an F which is linear, independent of any finite number of terms in the sequence, and satisfies $F[\langle \xi_n \rangle] \leq \limsup_{n\to\infty} \xi_n$ for every sequence.

It's interesting to note that any such F is an element of $\beta(\mathbb{N})$. Well it's interesting to me at least. I just want to tell you how I'm feeling. Got to make you understand.

10, 22. *Show that a Banach space X is reflexive iff X^* is reflexive. [Hint: If $\varphi[X]$ is not all of X^{**}, then there is a nonzero function $y \in X^{***}$ such that $y(x) = 0$ for all $x \in \varphi[X]$.]*

As with a lot of this stuff, the hardest part is sorting out exactly what you're talking about. This becomes somewhat easier if you clearly define the correspondences you need. To reduce confusion a little, I'll use the notation $x \in X$, $f \in X^*$, $\eta \in X^{**}$, and $\alpha \in X^{***}$.

So let $\varphi\colon X \to X^{**}$ is the map defined by $\varphi(x)(f) = f(x)$ for $x \in X$ and $f \in X^*$. Let $\psi\colon X^* \to X^{***}$ be the similar map defined by $\psi(f)(\eta) = \eta(f)$ for $f \in X^*$ and $\eta \in X^{**}$. We want to show that φ is onto if and only if ψ is.

First note that there exists another linear map $U\colon X^{***} \to X^*$ given by

$$U(\alpha)(x) = \alpha\big(\varphi(x)\big)$$

for $x \in X$ and $\alpha \in X^{***}$. So we can ask what $U \circ \psi\colon X^* \to X^*$ is: for any $x \in X$ and $f \in X^*$, we have

$$U(\psi(f))(x) = \psi(f)\big(\varphi(x)\big) = \varphi(x)(f) = f(x).$$

So $U \circ \psi$ is the identity.

On the other hand we have for any $\alpha \in X^{***}$ and any $\eta \in X^{**}$ that

$$\psi(U(\alpha))(\eta) = \eta\big(U(\alpha)\big)$$

which doesn't simplify immediately, and hence here's where the trouble might be.

If φ is actually an isomorphism onto X^{**}, then any $\eta \in X^{**}$ can be written as $\eta = \varphi(x)$ for some $x \in X$. In that case,

$$\eta\big(U(\alpha)\big) = \varphi(x)\big(U(\alpha)\big) = U(\alpha)(x) = \alpha\big(\varphi(x)\big) = \alpha(\eta),$$

so that $\psi \circ U$ is the identity. Hence in this case ψ is an isomorphism of X^* onto X^{***}.

Now suppose φ is *not* an isomorphism onto X^{**}. Then there is an $\eta \in X^{**}$ such that $\eta \notin \varphi[X]$. Assume without loss of generality that $\|\eta\|_{X^{**}} = 1$. Let $\{\eta\}$ be the space of multiples of η, and let $S \subset X^{**}$ be the subspace $\{\eta\} \oplus \varphi[X]$. Define a functional $\alpha \colon S \to \mathbb{R}$ by $\alpha(r\eta + w) = r$ for any $w \in \varphi[X]$, $r \in \mathbb{R}$. Then $\|\alpha\|_{S^*} = 1$ and we can extend α by the Hahn-Banach theorem to an element of $\widetilde{\alpha} \in X^{***}$ such that $\|\widetilde{\alpha}\|_{X^{***}} = 1$.

And remember, we want to prove that $\psi \circ U$ is not the identity in this case, so it will be sufficient to show that $\psi(U(\widetilde{\alpha})) \neq \widetilde{\alpha}$. To do this, we observe that for any $x \in X$, we know by definition of U that

$$U(\widetilde{\alpha})(x) = \widetilde{\alpha}(\big(\varphi(x)\big) = \alpha(\big(\varphi(x)\big)) = 0,$$

by the way we defined α. Since this is true for all x, $U(\widetilde{\alpha}) \equiv 0 \in X^*$.

Hence we have $\psi(U(\widetilde{\alpha})) = \psi(0) = 0 \neq \widetilde{\alpha}$, whence we conclude $\psi \circ U$ is *not* the identity.