# Math 6320 Homework #6 Solution

9, 41. *Let* $\mathbb{N}$ *be the set of natural numbers. Discuss* $\beta(\mathbb{N})$. *Show that a sequence from* $\mathbb{N}$ *converges in* $\beta(\mathbb{N})$ *if and only if it converges in* $\mathbb{N}$. *Hence* $\beta(\mathbb{N})$ *is compact but not sequentially compact.*

First of all, $\mathcal{F}$ is the set of all sequences $\langle x_n \rangle$ such that $0 \le x_n \le 1$ for all $n$. So $I^{\mathcal{F}}$ is the set of all functions from sequences in $[0,1]$ to numbers in $[0,1]$, in the topology of pointwise convergence. Intuitively $f(\langle x_n \rangle) = f(x_1, x_2, x_3, \ldots)$ is a function of infinitely many variables all in $[0,1]$.

The evaluation function $E_k$ is $E_k\langle x_n \rangle = x_k$. So $\beta(\mathbb{N})$ is the closure of $b(\mathbb{N}) \equiv \{E_k \,|\, k \in \mathbb{N}\}$ in $I^{\mathcal{F}}$ in the topology of pointwise convergence. Let's see what this is.

First observe that $b(\mathbb{N})$ is a discrete set in $I^{\mathcal{F}}$, since $\mathbb{N}$ has the discrete topology and is homeomorphic to $b(\mathbb{N})$.

Take any $G \in \beta(\mathbb{N})$, and any subbasis element containing $G$. By definition, this is the set of functions $F$ such that, for some fixed sequence $\langle y_n \rangle$ and some $\varepsilon > 0$, we have $|F(y_1, y_2, y_3, \ldots) - G(y_1, y_2, y_3, \ldots)| < \varepsilon$. We must have $E_k$ in this set for some $k$, by definition of closure. Hence for any fixed sequence $\langle y_n \rangle$ and any $\varepsilon > 0$ there is at least one $k$ such that $|G(y_1, y_2, y_3, \ldots) - y_k| < \varepsilon$. This implies that the value of $G(y_1, y_2, y_3, \ldots)$ is a point of closure of the set $\{y_1, y_2, y_3, \ldots\}$ (note: not necessarily a limit of a subsequence).

This narrows down the possibilities for $G \in \beta(\mathbb{N})$ significantly. We now just need to check the same thing for basis elements (so far we've only checked subbasis elements). The only difference is using the intersection of finitely many such sets; in other words, the condition is that for any finite set of sequences $\langle y_{1n} \rangle, \ldots, \langle y_{mn} \rangle$ and any $\varepsilon > 0$, there is a *single* $k$ such that $|G(y_{11}, y_{12}, \ldots) - y_{1k}| < \varepsilon, \ldots, |G(y_{m1}, y_{m2}, \ldots) - y_{mk}| < \varepsilon$.

Unfortunately this is a rather difficult problem; no explicit element $G$ satisfying this condition is actually known. On the other hand we know that there must be at least one, since $\mathbb{N}$ is not compact yet is homeomorphic to a subset of $\beta(\mathbb{N})$.

But moving on, let's show that a sequence from $\mathbb{N}$ converges in $\beta(\mathbb{N})$ if and only if it converges in $\mathbb{N}$. First we need to understand convergence of sequences: a sequence $F_k$ of maps from the space of sequences to reals converges to $G$ if and only if, for every fixed sequence $\langle y_n \rangle$, we have $\lim_{k\to\infty} F_k(y_1, y_2, y_3, \ldots) = G(y_1, y_2, y_3, \ldots)$. In particular if $E_n$ is the image of $n \in \mathbb{N}$, and $n_k$ is a sequence in $\mathbb{N}$, then $E_{n_k}$ converges to $G$ if and only if for every fixed sequence $\langle y_n \rangle$, we have $\lim_{k\to\infty} y_{n_k}$ existing.

If $\langle n_k \rangle$ is not eventually constant, then either a subsequence goes to infinity or a subsequence oscillates between two integers, and in either case we can choose $\langle y_n \rangle$ to consist of zero and one alternating so that this subsequence $y_{n_k}$ does not converge. Hence the only way $E_{n_k}$ converges is if $\langle n_k \rangle$ is eventually constant, which means that already $n_k$ converges to that constant in $\mathbb{N}$.

We conclude that no subsequence of $\langle E_n \rangle$ converges in $\beta(\mathbb{N})$, since no subsequence of $\langle n \rangle$ converges in $\mathbb{N}$. So $\beta(\mathbb{N})$ is not sequentially compact.

9, 42. *Let $f$ be a continuous periodic real-valued function on $\mathbb{R}$ with period $2\pi$; that is, $f(x + 2\pi) = f(x)$. Show that, given $\epsilon > 0$, there is a finite Fourier series $\varphi$, given by $\varphi(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$, such that $|\varphi(x) - f(x)| < \epsilon$ for all $x$. [Hint: Note that periodic functions are really functions on the circumference of the unit circle, and that $\cos mx \cos nx = \frac{1}{2}\{\cos(m+n)x - \cos(m-n)x\}$, etc.]*

First we observe that a periodic function $f : \mathbb{R} \to \mathbb{R}$ is equivalent to a function $\tilde{f} : S^1 \to \mathbb{R}$ under the quotient identification $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$, since the function respects equivalence classes. For continuity, it is enough to observe that the function $\tilde{f}$ is continuous when restricted to $(0, 2\pi)$ and $(-\pi, \pi)$, and thus by Proposition 8.3 $\tilde{f}$ is continuous on $S^1$. Of course the same thing works for the basis functions $\cos nx$ and $\sin nx$.

Clearly $S^1$ is compact, so we will have the result as long as we prove that $A$, the linear span of $\{\sin nx, \cos nx | n \in \mathbb{Z}\}$, is an algebra which separates points. That it is an algebra comes from the fact that for any

$m$ and $n$, we have

$$\begin{aligned}\cos mx \cos nx &= \frac{1}{2}\{\cos (m+n)x - \cos (m-n)x\},\\ \cos mx \sin nx &= \frac{1}{2}\{\sin (m+n)x - \sin (m-n)x\},\\ \sin mx \sin nx &= \frac{1}{2}\{\cos (m-n)x - \cos (m+n)x\}.\end{aligned}$$

So the product of linear combinations of these basis elements is also a linear combination of them.

For separating points, we take any $p$ and $q$ in $\mathbb{R}$. Let $f(x) = \cos(x-p) = \cos p \cos x - \sin p \sin x$. Then $f(p) = 1$, and if $f(q) = 1$ as well, then $\cos(q-p) = 1$, so $q - p = 2m\pi$ for some $m \in \mathbb{Z}$. Thus the projections satisfy $\tilde{p} = \tilde{q}$ in $S^1$.

10, 7. *Show that the set $P$ of all polynomials on $[0,1]$ is a linear manifold in $C[0,1]$. Is it closed? Give an example of a closed linear manifold in $C[0,1]$.*

Take any polynomials $P_1(x) = \sum_{k=0}^{n} a_k x^k$ and $P_2(x) = \sum_{k=0}^{m} b_k x^k$; then extending $a_k$ or $b_k$ by zeros up to $k = \max\{m,n\}$, we have

$$\lambda_1 P_1(x) + \lambda_2 P_2(x) = \sum_{k=0}^{\max\{m,n\}} (\lambda_1 a_k + \lambda_2 b_k) x^k.$$

Easy enough.

It is not closed, since by Corollary 9.35 (the Stone-Weierstrass theorem for polynomials on $\mathbb{R}$), we know $\overline{P} = C[0,1]$.

A simple example of a closed linear manifold is the set of lines:

$$\mathcal{L} = \{x \mapsto ax | a \in \mathbb{R}\}.$$

If any sequence $x \mapsto a_n x$ in $\mathcal{L}$ is Cauchy, then the coefficients $a_n$ must also be Cauchy, and so converges. Hence $\mathcal{L}$ is complete, and thus closed.

10, 13. *Show that if $A_n \to A$ and $x_n \to x$, then $A_n x_n \to Ax$.*

We have

$$\begin{aligned}\|A_n x_n - Ax\| &= \|(A_n - A)x_n + A(x_n - x)\| \\ &\leq \|(A_n - A)x_n\| + \|A(x_n - x)\| \\ &\leq \|(A_n - A)\|\|x_n\| + \|A\|\|x_n - x\|.\end{aligned}$$

Hence since we're no strangers to love (you know the rules and so do I), we can make the right side as small as we want.