Math 6320 Homework #5 Solution

9, 4. *Prove that a compact Hausdorff space is normal.*

This is easier if we first prove that it is regular. So let $F$ be a closed subset of a compact Hausdorff space and $y$ a point not in $F$. Then $F$ is also compact by Proposition 9.2. Since $X$ is Hausdorff, for each $x \in F$ there are disjoint open sets $U_x$ and $V_x$ such that $x \in U_x$ and $y \in V_x$.

Then $F \subset \cup_{x \in F} U_x$, so there is a finite set $\{x_1, \ldots, x_m\}$ such that if $O_1 = U_{x_1} \cup \cdots \cup U_{x_m}$, then $F \subset O_1$. Now we observe that $O_2 = V_{x_1} \cap \cdots \cap V_{x_m}$ is an open set containing $y$ and disjoint from $U_{x_1} \cup \cdots \cup U_{x_m}$. Hence we have found disjoint open sets containing $F$ and $y$, so $X$ is regular.

Using this we can prove that $X$ is normal. Let $F_1$ and $F_2$ be two disjoint closed sets. By regularity, for each $y \in F_2$, there are disjoint open sets $C_y$ and $D_y$ such that $F_1 \subset C_y$ and $y \in D_y$. Then the family $\cup_{y \in F_2} D_y$ is an open cover of $F_2$, and since $F_2$ is compact, there is a finite subcover $G_2 = D_{y_1} \cup \cdots \cup D_{y_n}$ of $F_2$. Then $F_1$ is contained in $G_1 = C_{y_1} \cap \cdots \cap C_{y_n}$, and $G_1$ and $G_2$ are disjoint.

9, 9c. *Let $\langle f_n \rangle$ be a decreasing sequence of upper semicontinuous functions which converge pointwise to a real-valued function $f$. Then $f$ is upper semicontinuous.*

We need to prove that for any $\alpha$, the set $f^{-1}(-\infty, \alpha)$ is open. So fix an $\alpha$.

For every $x$, we have $f(x) \leq f_n(x)$ for every $n \in \mathbb{N}$. Thus if $f_n(x) < \alpha$ then also $f(x) < \alpha$; in other words, $f_n^{-1}(-\infty, \alpha) \subset f^{-1}(-\infty, \alpha)$ for every $n$. So for every $\alpha$,

$$\bigcup_{n=1}^{\infty} f_n^{-1}(-\infty, \alpha) \subset f^{-1}(-\infty, \alpha).$$

Also we know that for every $x$, $\lim_{n \to \infty} f_n(x) = f(x)$. So if $f(x) < \alpha$, then there is an $N_x$ such that for every $n \geq N_x$, $|f_n(x) - f(x)| < \alpha - f(x)$; in other words, since $f_n(x) \geq f(x)$, we know $f_n(x) < \alpha$ for $n \geq N_x$. In particular $x \in f^{-1}(-\infty, \alpha)$ implies that $x \in f_{N_x}^{-1}(-\infty, \alpha)$. Hence

$$f^{-1}(-\infty, \alpha) \subset \bigcup_{n=1}^{\infty} f_n^{-1}(-\infty, \alpha).$$

We conclude that $f^{-1}(-\infty,\alpha) \subset \bigcup_{n=1}^{\infty} f_n^{-1}(-\infty,\alpha)$, and hence as a union of open sets, $f^{-1}(-\infty,\alpha)$ is also open.

9, 22a. *Let $O$ be an open subset of a compact Hausdorff space. Then $O$ is locally compact.*

Let $x \in O$; we want to find a set $U$ such that $x \in U \cap O$ and $\overline{U} \cap O$ is compact.

By regularity (see problem 4), since $\widetilde{O}$ is closed and $x \notin \widetilde{O}$, there are disjoint open sets $U$ and $V$ such that $x \in U$ and $\widetilde{O} \subset V$. In other words, $\widetilde{V} \subset O$, and $U \subset \widetilde{V}$.

Now $\widetilde{V}$ is closed in $X$, and hence also compact. Since $U \subset \widetilde{V}$, we know $\overline{U} \subset \widetilde{V}$, and hence $\overline{U}$ is compact. In addition $\overline{U} \subset O$, so in particular $\overline{U} \cap O$ is also compact (i.e., $\overline{U}$ is compact in the subspace topology of $O$).

9, 30. *Show that the function $\varphi$ constructed in the proof of Theorem 21 is continuous and proper.*

To prove continuity, it is enough to prove that for any $x$, there is an open $O \ni x$ such that the sum $\varphi(x) = \sum_{n=1}^{\infty}(1-\varphi_n(x))$ is actually a finite sum on $O$. Since $X = \cup O_n$, there is some $N$ such that $x \in O_N$. Now each $\varphi_n$ is identically 1 on $\overline{O}_{n-1}$, and since $O_N \subset O_n$ for every $n \geq N$, we know that whenever $n > N$, the function $(1-\varphi_n)$ is identically zero on $O_N$. Hence on $O_N$ we have $\varphi|_{O_N} = \sum_{n=1}^{N}(1-\varphi_n)|_{O_N}$. As a finite sum of continuous functions, $\varphi|_{O_N}$ is continuous, and since this is true for every $O_N$, the function $\varphi$ must be continuous.

To prove properness, we want to prove that $\varphi^{-1}[K]$ is a compact set in $X$ for each compact set $K$ in $(0,\infty)$. Now a subset of $(0,\infty)$ is compact if and only if it is closed and contained in $(0,M]$ for some positive integer $M$. Now $\varphi^{-1}[K]$ is a closed subset of $\varphi^{-1}(0,M]$, so it is enough to prove that $\varphi^{-1}(0,M]$ is compact in $X$.

Observe that for any $m$, if $x \notin O_m$ then for $n \leq m$ we know $x \notin O_n$, since $O_n \subset O_m$. Hence $\varphi_n(x) = 0$ since support $\varphi_n \subset O_n$. So if $x \notin O_m$ then

$$\varphi(x) = \sum_{n=1}^{\infty}(1-\varphi_n(x)) \geq \sum_{n=1}^{m}(1-\varphi_n(x)) = m.$$

Hence $\widetilde{O_m} \subset \varphi^{-1}[m, \infty)$, and we conclude that $\varphi^{-1}(0, m) \subset O_m$, and thus $\varphi^{-1}(0, m] \subset \overline{O}_m$, which is compact.