Math 6320 Homework #4 Solution

8, 1. (a) *Given a set X, can you define a metric on X so that the associated topological space is discrete? Trivial?*

The discrete metric, by an amazing coincidence, defines the discrete topology: set $\rho(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. Then balls of radius 1 are singletons, and hence every set is open (being a union of singletons).

The trivial topology cannot be described by a metric. If we have two points x and y, then the sequence $\{x, x, x, \ldots\}$ converges to x but also to y, since every set containing either one also contains all terms of the sequence. But in a metric space, limits are unique.

(b) *Let X be a space with a trivial topology. Find all continuous mappings of X into $\mathbb{R}$.*

A map is continuous if and only if $f^{-1}(U)$ is open whenever U is open in $\mathbb{R}$. If f is constant, so there is a $c \in \mathbb{R}$ such that $f(x) = c$ for all x, then $f^{-1}(U)$ is either X or $\emptyset$, depending on whether $c \in U$ or not.

Suppose f is not constant, and there are points x and y such that $f(x) \neq f(y)$. Choose ε so that $f(y) \notin B_\varepsilon(f(x))$. Then $f^{-1}[B_\varepsilon(f(x))]$ is a set containing x and not y, so it is not open. So f cannot be continuous.

(c) *Let X be a space with a discrete topology. Find all continuous mappings of X into $\mathbb{R}$.*

If $f\colon X \to \mathbb{R}$ is any function at all, then whenever $U \subset \mathbb{R}$ is open, $f^{-1}(U)$ is a set. Hence it is open. So *every* map is continuous.

8, 11. (a) *Let $\mathcal{B}$ be a base for the topological space $\langle X, \mathcal{J} \rangle$. Then $x \in \overline{E}$ if and only if for every $B \in \mathcal{B}$ with $x \in B$, there is a $y \in B \cap E$.*

First suppose $x \in \overline{E}$; then for every open $O \ni x$, we know $E \cap O \neq \emptyset$. In particular if $B \in \mathcal{B}$ with $B \ni x$, we know $E \cap B \neq \emptyset$.

Now suppose that for every $B \in \mathcal{B}$ with $x \in B$, there is a $y \in B \cap E$. Choose any open set $O \ni x$, and select a $B \in \mathcal{B}$ such that $x \in B \subset O$. Then $y \in B \cap E$, hence $O \cap E$ is not empty. Since O was arbitrary, x is a point of closure of E.

(b) *Let X satisfy the first axiom of countability. Then $x \in \overline{E}$ if and only if there is a sequence from E that converges to x.*

Let $\mathcal{B}_x$ be a countable family of open sets forming a basis at x. Define $C_n = B_1 \cap \cdots \cap B_n$ for every n; then $C_{n+1} \subset C_n$ for every n. If $x \in \overline{E}$, then for each n, the set C_n is open and contains x, so there is an $x_n \in C_n$. This sequence converges to x since for any open set $O \ni x$, we can find N such that $x \in B_N \subset O$. Hence also $C_N \subset O$, and for every $n \geq N$, we have $x_n \subset C_n \subset C_N \subset O$. The converse is obvious.

(c) *Let X satisfy the first axiom of countability. Then x is a cluster point of a sequence $\langle x_n \rangle$ from X if and only if $\langle x_n \rangle$ has a subsequence that converges to x.*

By definition, x is a cluster point of $\langle x_n \rangle$ if for every open $O \ni x$ and every $N \in \mathbb{N}$ there is an $k \geq N$ such that $x_k \in O$. Obviously if there is a convergent subsequence, we have a cluster point, so we just need to prove the other way around.

So choose nested sets $\langle C_n \rangle$ as in part (b), and for each $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $x_{n_k} \in C_k$. Clearly $\langle x_{n_k} \rangle$ converges to x.

8, 15. *Let X be the set of real numbers, and let $\mathcal{B}$ be the set of all intervals of the form $[a, b)$. Show that $\mathcal{B}$ is the base of a topology $\mathcal{J}$ for X. (This topology is called the half-open interval topology.) Show that $\langle X, \mathcal{J} \rangle$ satisfies the first but not the second axiom of countability and that the rationals are dense in X. Is $\langle X, \mathcal{J} \rangle$ metrizable?*

First check $\mathcal{B}$ is a base. By Proposition 8.5, we only need that $\mathcal{B}$ covers and that for every $x \in B_1 \cap B_2$ there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Obviously $\mathcal{B}$ covers since $x \in [x, x+1)$ for every x. Now if $[a, b) \cap [c, d)$ is nonempty, then $x < \min\{b, d\}$ and $x \geq \max\{a, c\}$. So $\big[\max\{a, c\}, \min\{b, d\}\big)$ is a B_3 that works.

Next show that $\mathcal{J}$ satisfies first countability. Let $x \in \mathbb{R}$ be any point, and let $V_{n,x} = [x, x + \frac{1}{n})$. Then each $V_{n,x}$ is open and contains x. Furthermore if O is any open set containing x, then there is a basis element $[a, b)$ such that $x \in [a, b) \subset O$ by definition of basis. Since $b > x$, there is an n such that $\frac{1}{n} < b - x$, and hence for that n we know $V_{n,x} \subset [a, b) \subset O$. So $V_{n,x}$ is a countable basis at x.

Now show that $\mathcal{J}$ does not satisfy second countability. Suppose $\mathcal{C}$ is any basis for the topology. For each $C \in \mathcal{C}$ let $i(C) = \inf C$. Then for each $x \in \mathbb{R}$ the set $[x, x+1)$ is open and contains x, so there is a $C_x \in \mathcal{C}$ such that $x \in C_x \subset [x, x+1)$. Clearly $i(C_x) = x$. So the image of $i[\mathcal{C}]$ contains all real numbers, and hence $\mathcal{C}$ must be uncountable.

After all that we show the rationals are dense. Let O be any open set; then O contains a basis element $[a, b)$ for some $b > a$. There is a rational $q \in (a, b)$, so $O \cap \mathbb{Q}$ is not empty.

Finally, we observe that $\langle X, \mathcal{J} \rangle$ cannot be metrizable, since separability and second countability are equivalent for metric spaces.

8, 18. (a) *Show that every metric space is Hausdorff.*

Let x and y be distinct points, and let $\varepsilon = \frac{1}{2}\rho(x, y)$. Then $\varepsilon > 0$ so $O_1 = B_\varepsilon(x)$ and $O_2 = B_\varepsilon(y)$ contain x and y respectively. Since $\rho(x, z) + \rho(y, z) \geq \rho(x, y) = 2\varepsilon$, there is no z in both O_1 and O_2, so they are disjoint.

(b) *Show that every metric space is normal. [Hint: If F_1 and F_2 are disjoint closed sets, let $O_1 = \{x : \rho(x, F_1) < \rho(x, F_2)\}$ and $O_2 = \{x : \rho(x, F_2) < \rho(x, F_1)\}$.]*

Following the hint, we just need to show O_1 and O_2 are open, that they are disjoint, and that $F_1 \subset O_1$ and $F_2 \subset O_2$.

To show O_1 is open, let $x \in O_1$, and choose $\varepsilon = (\rho(x, F_2) - \rho(x, F_1))/3 > 0$. Suppose $z \in B_\varepsilon(x)$. Then for any $y \in F_1$, we have $\rho(z, y) \leq \rho(x, y) + \rho(x, z) < \rho(x, y) + \varepsilon$, and thus taking infima over y we get $\rho(z, F_1) \leq \rho(x, F_1) + \varepsilon$. On the other hand $\rho(x, F_2) \leq \rho(z, F_2) + \varepsilon$ by the same reasoning, so $\rho(z, F_2) \geq \rho(x, F_2) - \varepsilon$. Therefore we have

$$\rho(z, F_2) - \rho(z, F_1) \geq \rho(x, F_2) - \varepsilon - (\rho(x, F_1) - \varepsilon = 3\varepsilon - 2\varepsilon = \varepsilon > 0.$$

Thus $z \in O_1$. Since $z \in B_\varepsilon(x)$ was arbitrary, never gonna give you up, never gonna let you down, never gonna run around and desert you. Similarly O_2 is open.

That they are disjoint is obvious: there's no way both inequalities could be valid simultaneously. Finally, if $y \in F_1$ then $\rho(y, F_1) = 0$. On the other hand $\rho(y, F_2)$ must be positive; if it were zero, then we could find a sequence $\langle z_n \rangle$ in F_2 converging to y, which would force $y \in F_2$ as well since F_2 is closed. So $\rho(y, F_2) > \rho(y, F_1)$ and thus $y \in O_1$. We conclude $F_1 \subset O_1$, and similarly $F_2 \subset O_2$.

8, 26. *Let $\mathcal{F}$ be a family of real-valued continuous functions on a topological space $\langle X, \mathcal{J} \rangle$. Show that the weak topology generated by $\mathcal{F}$ is $\mathcal{J}$ if for each closed set F and each $x \notin F$ there is an $f \in \mathcal{F}$ with $f(x) = 1$ and $f \equiv 0$ on F.*

Any set which is weak-open must be of the form

$$W = \bigcup_{\alpha} \bigcap_{i=1}^{n} f_{i,\alpha}^{-1}(O_{i,\alpha})$$

for some continuous functions $f_{i,\alpha}$ and open real subsets $O_{i,\alpha}$, by definition of the weak topology. So a weak-open set is also strong-open.

We just need to prove that a strong-open set is also weak-open. So let V be a strong-open set; then $\widetilde{V}$ is closed, and so for each $y \in V$ we can find a continuous function $f_y \colon X \to \mathbb{R}$ such that $f_y(y) = 1$ and $f_y \equiv 0$ on $\widetilde{V}$. Thus if $U = \mathbb{R} \backslash \{0\}$, then U is open and $O_y = f_y^{-1}(U)$ is an open set which contains y and is disjoint from $\widetilde{V}$, i.e., completely contained in V.

Hence

$$V = \bigcup_{y \in V} O_y = \bigcup_{y \in V} f_y^{-1}(U),$$

which is exactly the form of a weak-open set.