

Math 6320 Homework #3 Solution

- 7, 32. 1. *Prove that, if E is of first category and $A \subset E$, then A is also of first category.*

We have $E = \bigcup_{n=1}^{\infty} E_n$ where every E_n is a nowhere dense set. Hence

$$A = A \cap E = A \cap \left(\bigcup_{n=1}^{\infty} E_n \right) = \bigcup_{n=1}^{\infty} (A \cap E_n).$$

Since each $A \cap E_n$ is nowhere dense, A is first category.

2. *Prove that if $\langle E_n \rangle$ is a sequence of sets of first category, then $\bigcup E_n$ is also of first category.*

Let $E_m = \bigcup_{n=1}^{\infty} E_{mn}$ where every E_{mn} is nowhere dense. Then

$$E = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{mn} = \bigcup_{m,n=1}^{\infty} E_{mn}.$$

Enumerating $\mathbb{N} \times \mathbb{N}$ in the standard way, we get that E is a countable union of nowhere dense sets.

- 7, 33. 1. *Show that on $[0, 1]$ there is a nowhere dense closed set having Lebesgue measure $1 - 1/n$.*

The easiest way to do this is to find a dense open set whose measure is $\frac{1}{n}$; then its complement will be a nowhere dense closed set of measure $1 - \frac{1}{n}$.

So count the rationals in $[0, 1]$ as q_1, q_2, \dots . For each $n > 1$ and m let $U_{nm} = (q_m - \frac{1}{n2^{m+1}}, q_m + \frac{1}{n2^{m+1}})$. Then for each n , the set $\bigcup_{m=1}^{\infty} U_{nm}$ has measure at most $\sum_{m=1}^{\infty} \frac{1}{n2^m} = \frac{1}{n}$. It is clearly open as a union of open intervals, and it is clearly dense as it contains all the rationals. The only drawback is that there is probably some overlap, and the measure is not *exactly* equal to $\frac{1}{n}$, just bounded above by it. One could probably find a larger open set of measure exactly equal to $\frac{1}{n}$, but it's likely easier simply to use the generalized Cantor set from problem 3.14.

2. *Construct a set of first category on $[0, 1]$ that has measure 1.*

For this part, it doesn't matter which version we use. We take E_n to be a nowhere dense closed set of measure at least $1 - \frac{1}{n}$, and

let $E = \bigcup_{n=1}^{\infty} E_n$. Then E is a first category set by definition. The measure of E is at least as large as the measure of any E_n ; in particular it is at least $1 - \frac{1}{n}$ for every n . So it must be 1.

7, 37. Let E be a subset of a complete metric space.

1. If \tilde{E} is dense and F a closed set contained in E , then F is nowhere dense.

We have $F \subset E$ so that $\tilde{E} \subset \tilde{F}$. Now \tilde{E} is dense, and since \tilde{F} contains it, \tilde{F} is also dense. But since F is closed, $\overline{\tilde{F}} = F$, so \tilde{F} is dense. Thus by definition F is nowhere dense.

2. If E and \tilde{E} are both dense, then at most one of them is an F_{σ} .

Suppose E and \tilde{E} are both dense, and assume that E and \tilde{E} are both F_{σ} . Then $\tilde{E} = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed, and therefore $E = \bigcap_{n=1}^{\infty} \tilde{F}_n$ where each \tilde{F}_n is open. Now E is dense and hence each F_n must be dense (since E is contained in all of them). That implies that each F_n is nowhere dense.

On the other hand, E is also an F_{σ} , so $E = \bigcup_{n=1}^{\infty} G_n$ where each G_n is closed, and exactly as above each G_n must be nowhere dense. But $X = E \cup \tilde{E}$, so X is equal to the union of all the F_n and all the G_n . Thus X is a countable union of nowhere dense sets, and this is impossible by the Baire category theorem.

3. The set of rational numbers in $[0, 1]$ is not a G_{δ} .

The rationals are an F_{σ} since they are a countable set. Since the irrationals are also dense, they cannot also be an F_{σ} . Hence the rationals cannot be a G_{δ} , since every G_{δ} is the complement of an F_{σ} .

4. Is there a real-valued function on $[0, 1]$ which is continuous on the rationals and discontinuous on the irrationals?

This sounds like a statement that relates to the famous factoid that the set of points at which an arbitrary function $f: [0, 1] \rightarrow \mathbb{R}$ is continuous is always a G_{δ} . Given this factoid, the answer to the question must be “no,” since the rationals are not a G_{δ} .

To prove the set of points at which a function is continuous is a G_{δ} (in case you’ve never seen it), observe that f is continuous at x if and only if for every $n \in \mathbb{N}$ there is a $\delta_x > 0$ such that $|y - z| < \delta_x$

is contained in $f^{-1}(f(x) - \frac{1}{n}, f(x) + \frac{1}{n})$. We need to rephrase this slightly, as the construction we're about to do won't work quite this way.

So observe that if f is continuous at x , then for any ε there is a δ such that $|y - x| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Now if y and z are both contained in $(x - \frac{\delta}{2}, x + \frac{\delta}{2})$, then $|y - x| < \frac{\delta}{2}$ and $|z - x| < \frac{\delta}{2}$, so $|y - z| < \delta$. Hence $|f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| < 2\varepsilon$. Thus replacing δ with $\delta/2$ and ε with $\frac{\varepsilon}{2}$, we can say that if f is continuous at x , then for every ε there is a $\tilde{\delta} > 0$ such that $y, z \in (x - \tilde{\delta}, x + \tilde{\delta})$ implies $|f(y) - f(z)| < \varepsilon$. Conversely, if this condition is true, then in particular taking $z = x$ we'd get the standard condition for continuity at x . So this is an alternative definition of continuity at x .

Now for each $n \in \mathbb{N}$, define

$$E_n = \left\{ x \in [0, 1] \mid \exists \delta_x > 0 : y, z \in B_{\delta_x}(x) \Rightarrow |f(y) - f(z)| < \frac{1}{n} \right\}.$$

Clearly the intersection of all E_n is exactly the set of points where f is continuous, so we'll be done if every E_n is open. So suppose $x \in E_n$ and choose δ_x satisfying the condition. We want to prove that $B_{\delta_x}(x)$ is actually contained in E_n . So let $w \in B_{\delta_x}(x)$ be any point and choose δ_w so that $B_{\delta_w}(w) \subset B_{\delta_x}(x)$. Then $y, z \in B_{\delta_w}(w)$ implies $y, z \in B_{\delta_x}(x)$ which implies $|f(y) - f(z)| < \frac{1}{n}$. This shows that $w \in E_n$, and hence $B_{\delta_x}(x) \subset E_n$, so each E_n is open.

- 7, 50. A real-valued function f on $[0, 1]$ is said to be Hölder continuous of order α if there is a constant C such that $|f(x) - f(y)| \leq C|x - y|^\alpha$. Define $\|f\|_\alpha = \max|f(x)| + \sup|f(x) - f(y)|/|x - y|^\alpha$.

Show that for $0 < \alpha \leq 1$, the set of functions with $\|f\|_\alpha \leq 1$ is a compact subset of $C[0, 1]$.

Denote by \mathcal{F} the set of f such that $\|f\|_\alpha \leq 1$. The space $C[0, 1]$ is a metric space under the usual supremum norm, so to prove a subset is compact we only need to prove it is sequentially compact. So take any sequence f_n of functions in \mathcal{F} ; we will construct a subsequence that converges uniformly to a limit in \mathcal{F} .

We verify the conditions of the Ascoli-Arzelá theorem. First $X = [0, 1]$ is separable since the rationals are dense. Since $|f_n(x)| \leq \|f_n\|_\alpha \leq 1$ for

every x and every n , we know the closure of $\{f_n(x) \mid n \in \mathbb{N}\}$ is contained in $[-1, 1]$, hence compact. Finally for equicontinuity, we observe that for every $x, y \in [0, 1]$ and $n \in \mathbb{N}$, we have $|f_n(x) - f_n(y)| \leq |x - y|^\alpha$. So given $\varepsilon > 0$ choose $\delta = \varepsilon^{1/\alpha}$; then $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon$ for every x, y, n . Hence the family is uniformly equicontinuous.

The Ascoli theorem then gives a subsequence that converges to a limit f uniformly on every compact set; in particular since $[0, 1]$ is compact, the sequence f_{n_k} converges uniformly to f on $[0, 1]$. Thus f must be continuous. Finally we have for every x that $|f(x)| \leq \limsup_{k \rightarrow \infty} |f_{n_k}(x)|$, and for every x and y that $|f(x) - f(y)| = \limsup_{k \rightarrow \infty} |f_{n_k}(x) - f_{n_k}(y)|$. Thus

$$\|f\|_\alpha \leq \limsup_{k \rightarrow \infty} \|f_{n_k}\| \leq 1.$$

- **Extra problem 1** *Is there a set of measure zero which is not meager?*

By problem 33, the complement of a meager set in $[0, 1]$ is not meager. So the complement of the measure-one set constructed in problem 37 has measure zero and is not meager.

- **Extra problem 2** *If $f(x) = \exp(-1/x^2)$ when x is not zero and $f(0) = 0$, prove carefully that f has infinitely many continuous derivatives everywhere. How could you use this fact to get a smooth function which is zero outside $[0, 1]$ and nonzero inside?*

By the Chain rule, when x is not zero we have $f'(x) = \frac{2}{x^3} \exp(-1/x^2)$, and it's easy to prove inductively that the derivatives of f will all have the form

$$f^{(n)}(x) = \sum_{k=1}^n c_{kn} x^{-n-2k} e^{-1/x^2};$$

we just verify that

$$\begin{aligned} f^{(n+1)}(x) &= \sum_{k=1}^n \left[(-n - 2k) c_{kn} x^{-n-1-2k} e^{-1/x^2} + 2c_{kn} x^{-n-3-2k} \right] \\ &= - \sum_{k=1}^n (n + 2k) c_{kn} x^{-(n+1)-2k} e^{-1/x^2} + \sum_{k=2}^{n+1} 2c_{kn} x^{-(n+1)-2k}. \end{aligned}$$

For x not equal to zero, these functions are obviously all continuous.

So all that needs to be checked is the smoothness at $x = 0$: for every n , we want to prove that $f^{(n)}(0)$ exists and that $\lim_{x \rightarrow 0} f^{(n)} = f^{(n)}(0)$. We first observe that for every m , the following limit is valid:

$$\lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^m} = 0.$$

The easiest way to see this is to redefine $y = 1/x^2$; then we have

$$\lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^m} = \lim_{y \rightarrow \infty} \frac{e^{-y}}{y^{-m/2}} = \lim_{y \rightarrow \infty} \frac{y^{m/2}}{e^y}.$$

We can then use either L'Hopital's rule or the more elementary fact that for any p , there is a $C_p > 0$ such that $e^y > C_p y^p$ for all $y > 0$. Either way, the limit must be zero.

This establishes in particular that

$$f'(0) = \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h} = 0.$$

Then inductively we compute $f^{(n)}(0)$ using the definition:

$$f^{(n+1)}(0) = \lim_{h \rightarrow 0} \frac{f^{(n)}(h)}{h} = \sum_{k=1}^n c_{kn} \lim_{h \rightarrow 0} h^{-n-2k-1} e^{-1/h^2} = 0.$$

This establishes the existence of all derivatives at $x = 0$ (and that they are all zero).

Finally we check that all the derivatives are actually continuous at $x = 0$: we have

$$\lim_{x \rightarrow 0} f^{(n)}(x) = \sum_{k=1}^n c_{kn} \lim_{x \rightarrow 0} x^{-n-2k} e^{-1/x^2} = 0.$$

Since all derivatives are zero at $x = 0$, we could break the function in half there and replace the left half with the constant function zero, and the resulting function

$$g(x) = \begin{cases} 0 & x \leq 0 \\ \exp(-1/x^2) & x > 0 \end{cases}$$

will still be C^∞ . and nowhere zero on $(0, \infty)$. Finally we could multiply to get $h(x) = g(x)g(1-x)$; the product of two C^∞ functions is C^∞ by the product rule, and clearly $h(x)$ is zero if and only if $x \notin [0, 1]$.