Math 6320 Homework #2 Solution

7, 21. We want to prove Proposition 7.11:

Let $\langle X, \rho \rangle$ and $\langle Y, \sigma \rangle$ be metric spaces with Y complete. Let f be a uniformly continuous mapping from a subset E of X into Y. Then there is a unique continuous extension g of f from E to $\overline{E}$; that is, there is a unique continuous mapping g from $\overline{E}$ into Y such that $g(x) = f(x)$ for $x \in E$. Moreover, g is uniformly continuous.

a. "If $\langle x_n \rangle$ is a sequence in E that converges to a point $x \in \overline{E}$, then $\langle f(x_n) \rangle$ converges to a point $y \in Y$."

Since $\langle x_n \rangle$ converges, it is a Cauchy sequence in E. Since f is uniformly continuous on E, $\langle f(x_n) \rangle$ is also a Cauchy sequence by Proposition 7.10. Since Y is complete, $\langle f(x_n) \rangle$ converges.

b. "The point y in (a) depends only on x and not on the sequence $\langle x_n \rangle$. Thus if we define $y = g(x)$, we have defined a function g on $\overline{E}$ which is an extension of f."

Suppose $\langle x_n \rangle$ and $\langle u_n \rangle$ are two sequences converging to the same $x \in \overline{E}$. Let $y = \lim_{n \to \infty} f(x_n)$ and $v = \lim_{n \to \infty} f(u_n)$. We want to show $y = v$.

Let $\varepsilon > 0$ be any number. By uniform continuity, choose δ so that $\rho(s,t) < \delta$ implies $\sigma(f(s), f(t)) < \frac{\varepsilon}{3}$ for any $s, t \in X$. Since $\langle x_n \rangle$ and $\langle u_n \rangle$ both converge to x, we have

$$\rho(x_n, u_n) \leq \rho(x_n, x) + \rho(x, u_n),$$

so that we can choose N_1 large enough that $n \geq N_1$ implies $\rho(x_n, u_n) < \delta$. We can also choose N_2 so that $n \geq N_2$ implies both $\sigma(f(x_n), y) < \frac{\varepsilon}{3}$ and also $\sigma(f(u_n), v) < \frac{\varepsilon}{3}$. Finally choosing $N = \max\{N_1, N_2\}$ we get

$$\begin{aligned} \sigma(y, v) &\leq \sigma(y, f(x_N)) + \sigma(f(x_N), f(u_N)) + \sigma(f(u_N), v) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since $\sigma(y, v) < \varepsilon$ for every $\varepsilon > 0$, we must have $\sigma(y, v) = 0$, so $y = v$.

c. "The function g is uniformly continuous on $\overline{E}$."

Let $\varepsilon > 0$. Choose δ so that for every $s, t \in E$ with $\rho(s,t) < \delta$, we have $\sigma(f(s), f(t)) < \frac{\varepsilon}{3}$. Let x and y be any points in $\overline{E}$ such that $\rho(x,y) < \frac{\delta}{3}$. We want to show $\sigma(g(x), g(y)) < \varepsilon$.

So let $\langle x_n \rangle$ be any sequence converging to $x \in \overline{E}$, and $\langle y_n \rangle$ any sequence converging to $y \in \overline{E}$. Choose N_1 so large that $\rho(x_n, x) < \frac{\delta}{3}$ and $\rho(y_n, y) < \frac{\delta}{3}$ for all $n \geq N_1$. Then for any $n \geq N_1$, we have

$$\rho(x_n, y_n) \leq \rho(x_n, x) + \rho(x, y) + \rho(y, y_n) < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.$$

Therefore we know

$$\sigma(f(x_n), f(y_n)) < \frac{\varepsilon}{3}$$

whenever $n \geq N_1$.

Now choose N_2 so that $n \geq N_2$ implies $\sigma(f(x_n), g(x)) < \frac{\varepsilon}{3}$ and also $\sigma(f(y_n), g(y)) < \frac{\varepsilon}{3}$. Then for any $n \geq \max\{N_1, N_2\}$, we have

$$\begin{aligned}\sigma(g(x), g(y)) &\leq \sigma(g(x), f(x_n)) + \sigma(f(x_n), f(y_n)) + \sigma(f(y_n), g(y)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since $n \geq \max\{N_1, N_2\}$ was arbitrary, and the only thing we required is that $\rho(x,y) < \frac{\delta}{3}$, we have proved uniform continuity of g.

d. "If h is any continuous function from $\overline{E}$ to Y that agrees with f on E, then $h \equiv g$."

Let $x \in \overline{E}$ be any point; we want to show $h(x) = g(x)$. Let $\langle x_n \rangle$ be any sequence in E that converges to x; then since g and h are both continuous, we have

$$h(x) = \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} f(x_n),$$

as well as

$$g(x) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n).$$

Hence since limits are unique, $h(x) = g(x)$.

7, 25. We want to prove Proposition 7.14:

If a subset A of a metric space is complete, then it is closed. On the other hand, a closed subset of a complete metric space is itself complete.

First we suppose X is a metric space and A is complete in X; that is, every Cauchy sequence in A converges in A. We want to prove A is closed. So let $x \in \overline{A}$ and choose a sequence $\langle x_n \rangle$ in A converging to x. Then $\langle x_n \rangle$ is a Cauchy sequence in X, since $\rho(x_m, x_n) \leq \rho(x_m, x) + \rho(x, x_n)$ for every m and n. Since all x_n are in A, we know $\langle x_n \rangle$ is also Cauchy in A. Since A is complete, we have $\lim_{n \to \infty} x_n = y$ for some $y \in A$. But since limits are unique, $y = x$, so $x \in A$ as well.

Now we suppose A is closed in X and X is complete. We want to prove A is complete. So let $\langle x_n \rangle$ be a Cauchy sequence in A; then obviously $\langle x_n \rangle$ is also a Cauchy sequence in X, so it converges in X to some $x \in X$. Since $\lim_{n \to \infty} x_n = x$, we know that for every $\varepsilon > 0$ that there is an N such that for every $n \geq N$, we have $\rho(x_n, x) < \varepsilon$. In particular there is at least one $y \in A$ such that $\rho(x, y) < \varepsilon$ for every ε. So x is a point of closure of A, and hence contained in A since A is closed.

7, 29. We want to prove Proposition 7.20:

Let $\mathcal{U}$ be an open cover of a (sequentially) compact metric space X. Then there is a number $\epsilon > 0$ such that for each $x \in X$ and each $\delta < \epsilon$ the ball $B_{x,\delta}$ is contained in some open set $O \in \mathcal{U}$.

a. "We need only consider the case when $X \notin \mathcal{U}$. Set

$$\varphi(x) = \sup\{r : \exists O \in \mathcal{U} \text{ with } B_{x,r} \subset O\}.$$

Show that $0 < \varphi(x) < \infty$."

The reason we can assume $X \notin \mathcal{U}$ is that if it were, then *every* ball $B_{x,\delta}$ is contained in X so ϵ could be any number at all and we'd be done immediately. So let's assume that.

To show $\varphi(x) > 0$, we just need to show the set of which it's the supremum is nonempty. This is certainly true; each $x \in X$ is contained in some open set $O \in \mathcal{U}$, and hence by definition of opennness there is some $r > 0$ such that $B_{x,r} \subset O$.

To show $\varphi(x) < \infty$, we need to show that there is a number M such that $r < M$ whenever $B_{x,r}$ is contained in an open set

O. Recalling that $f_x\colon X \to \mathbb{R}$ defined by $f_x(y) = \rho(x,y)$ is a continuous function, we know by Theorem 7.18 that f_x attains its maximum on X. So choose $D > \sup_{y\in X} f_x(y)$. Then $B_{x,D} = X$, so that $B_{x,D}$ is not contained in any open $O \in \mathcal{U}$. Hence $\varphi(x) \le D$, and in particular it is less than infinity.

b. "Show that for each x and y

$$\varphi(y) \ge \varphi(x) - \rho(x,y).''$$

The basic idea, roughly, is that if $B_{x,\varphi(x)} \subset O$ for some O, and if $y \in B_{x,\varphi(x)}$, then there is a small ball $B_{y,\delta}$ which fits in $B_{x,\varphi(x)}$ and hence in O, and thus $\varphi(y) \ge \delta$. If y is *not* in $B_{x,\varphi(x)}$, then $\varphi(x) - \rho(x,y) \le 0$ and the statement is trivially true.

This idea doesn't quite work, as we don't know $\varphi(x) \in \{r : \exists O \in \mathcal{U} \text{ with } B_{x,r} \subset O\}$, only that it's the supremum. But it's close enough.

So suppose $\varphi(x) - \rho(x,y) > 0$; if not then the statement is obvious. Let ε be any number such that $0 < \varepsilon < \varphi(x) - \rho(x,y)$. Then by definition of supremum, there is some r with $\varphi(x) - \varepsilon < r < \varphi(x)$ such that $B_{x,r} \subset O$ for some $O \in \mathcal{U}$. By construction we have

$$\rho(x,y) < \varphi(x) - \varepsilon < r,$$

so that $y \in B_{x,r}$. Hence there is a $\delta > 0$ such that $B_{y,\delta} \subset B_{x,r}$, and in fact we can estimate how large that δ is.

If $\rho(y,z) < \delta$, then we have

$$\rho(x,z) \le \rho(x,y) + \rho(y,z) < \delta + \rho(x,y).$$

So choosing $\delta = r - \rho(x,y)$ we have $\rho(y,z) < \delta$ implying $\rho(x,z) < r$, or in other words $B_{y,\delta} \subset B_{x,r} \subset O$. We conclude that $\varphi(y) \ge \delta$. Now since $r > \varphi(x) - \varepsilon$, we know that

$$\varphi(y) \ge r - \rho(x,y) > \varphi(x) - \rho(x,y) - \varepsilon.$$

Since this is true for any small positive ε, we must have

$$\varphi(y) \ge \varphi(x) - \rho(x,y).$$

c. "Show that φ is a continuous function on X."

We have from part (b) that

$$\varphi(x) - \varphi(y) \leq \rho(x, y)$$

for all x and y, and by symmetry we also have

$$\varphi(y) - \varphi(x) \leq \rho(x, y).$$

Therefore

$$|\varphi(x) - \varphi(y)| \leq \rho(x, y),$$

and φ is in fact uniformly continuous on X.

d. "If X is sequentially compact, then $\epsilon = \inf \varphi$ is positive."

By Theorem 18, φ attains its minimum on K. So $\epsilon = \varphi(p)$ for some p, and since $\varphi(p) > 0$ by (a), we are done.

e. "This ϵ satisfies the conditions of the proposition."

Let $x \in X$ and $\delta < \epsilon$. Then $\delta < \varphi(x)$, so there is some r with $\delta < r < \varphi(x)$ such that $B_{x,r} \subset O$ for some O. In particular $B_{x,\delta} \subset B_{x,r} \subset O$.