Math 6320 Homework #11 Solution

11, 42. *Prove Lemma 27 for the case $p = 1$: Let $(X, \mathcal{B}, \mu)$ be a finite measure space and g an integrable function such that for some constant M,*

$$|\int g\varphi \, d\mu| \le M\|\varphi\|_1$$

for all simple functions φ. Then $g \in L^\infty$.

Let $\varepsilon > 0$ and let $E = \{x \mid |g(x)| > M+\varepsilon\}$. We want to prove $\mu(E) = 0$; this will show that $\|g\|_{L^\infty} \le M + \varepsilon$.

To make things a bit easier, split E into the disjoint sets $E_+ = \{x \mid g(x) > M + \varepsilon\}$ and $E_- = \{x \mid g(x) < -M - \varepsilon\}$.

Let $\varphi = \xi_{E_+} - \xi_{E_-}$. Then φ is a simple function, so we have

$$|\int g\varphi \, d\mu| \le M|\varphi|_1 = M(\mu(E_+) + \mu(E_-)) = M\mu(E).$$

On the other hand we also have

$$\int g\varphi \, d\mu = \int_{E_+} g \, d\mu - \int_{E_-} g \, d\mu \ge (M+\varepsilon)\mu(E_+) + (M+\varepsilon)\mu(E_-) = (M+\varepsilon)\mu(E).$$

The only way we can have

$$(M + \varepsilon)\mu(E) \le M\mu(E)$$

is if $\mu(E) = 0$, which is what we wanted to show.

Since this is true for every ε, we have $\|g\|_{L^\infty} \le M$, and in particular $g \in L^\infty$.

12, 7. *Let μ be a finite measure on an algebra $\mathcal{A}$, and μ^* the induced outer measure. Show that a set E is measurable if and only if for each $\epsilon > 0$ there is a set $A \in \mathcal{A}_\delta$, $A \subset E$, such that $\mu^*(E\ A) < \epsilon$.*

First suppose E is measurable.

Clearly $\mathcal{A}_\delta$ is the set of complements of sets in $\mathcal{A}_\sigma$, since

$$E = \cap_{n=1}^\infty E_n \Leftrightarrow \widetilde{E} = \cup_{n=1}^\infty \widetilde{E_n}$$

and $\mathcal{A}$ is closed under complements. So the idea is to use Proposition 6 on complements.

Proposition 6 says that for *any* set E and any $\epsilon > 0$, there is a set $A \in \mathcal{A}_\sigma$ with $E \subset A$ and

$$\mu^* A \leq \mu^* E + \epsilon.$$

Hence for any set E, there is a set $B \in \mathcal{A}_\sigma$ with $\widetilde{E} \subset B$ and

$$\mu^* B \leq \mu^*(\widetilde{E}) + \epsilon. \tag{1}$$

Then $A = \widetilde{B} \in \mathcal{A}_\delta$ and

$$\mu^*(\widetilde{A}) \leq \mu^*(\widetilde{E}) + \epsilon.$$

Now μ is finite, so that $\mu^*(X) < \infty$, and since A is measurable, we have

$$\mu^*(X) = \mu^*(X \cap A) + \mu^*(X \cap B) = \mu^*(A) + \mu^*(B).$$

Thus by Proposition 6 we get

$$\mu^*(X) - \mu^*(A) \leq \mu^*(\widetilde{E}) + \epsilon. \tag{2}$$

Now if E is measurable then

$$\mu^*(X) = \mu^*(E) + \mu^*(\widetilde{E}),$$

and thus equation (2) becomes

$$\mu^*(E) \leq \mu^*(A) + \epsilon.$$

Finally since A is measurable and $A \subset E$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap \widetilde{A}) = \mu^*(A) + \mu^*(E \cap \widetilde{A}).$$

Combining this with the previous inequality we have

$$\mu^*(E \cap \widetilde{A}) \leq \epsilon.$$

Now let's prove the converse: suppose E is a set such that for any $\epsilon > 0$ there is a set $A \subset E$, $A \in \mathcal{A}_\delta$, such that $\mu(E \cap \widetilde{A}) \leq \epsilon$. We want to prove that E is measurable. The easiest way to do this is to prove that E is a countable union of elements of $\mathcal{A}_\delta$, since we know that the measurable sets are a σ-algebra.

Having decided this, there's only one thing to do: for each $n \in \mathbb{N}$ let $\epsilon = \frac{1}{n}$, and choose $A_n \in \mathcal{A}_\delta$ such that $A_n \subset E$ and $\mu^*(E \cap \widetilde{A_n}) \leq \frac{1}{n}$. Let $A = \cup_{n=1}^{\infty} A_n$. We know that A is measurable since it's a countable union of countable intersections of measurable sets. If we can prove $\mu^*(E \cap \widetilde{A}) = 0$, then E will be a measurable set, since any set with outer measure zero is measurable.

Since $A_n \subset A$ for every n we know $\widetilde{A} \subset \widetilde{A_n}$ so that $E \cap \widetilde{A} \subset E \cap \widetilde{A_n}$. Thus by monotonicity,

$$\mu^*(E \cap \widetilde{A}) \leq \mu^*(E \cap \widetilde{A_n}) \leq \frac{1}{n},$$

and since this is true for all n we have

$$\mu^*(E \cap \widetilde{A}) = 0.$$

So $E \cap \widetilde{A}$ is measurable, and A is measurable, and thus

$$E = (E \cap \widetilde{A}) \cup A$$

is measurable.

12, 8e. *Let X be a set consisting of two points. Construct an outer measure on X which is not regular.*

There are only four subsets of X: $\emptyset$, $C = \{x\}$, $D = \{y\}$, and $X = \{x, y\}$.

Let $\mu^*(\emptyset) = 0$, $\mu^*(C) = c$, $\mu^*(D) = d$, and $\mu^*(X) = x$. Subadditivity imposes the following constraints:

$$\begin{aligned} c &\leq x \\ d &\leq x \\ x &\leq c + d. \end{aligned}$$

Conversely any triplet of numbers $a, b, x \in [0, \infty]$ defines an outer measure on X.

The sets C and D are measurable if and only if

$$x = c + d.$$

If all sets are measurable then the outer measure is certainly regular (we can just choose $A = E$ every time), so we want $x < c + d$ for the counterexample.

To be specific, let $\mu^*(\{x\}) = 1$, $\mu^*(\{y\}) = 2$, and $\mu^*(\{x, y\}) = 2$.

To prove such an outer measure is not regular, let $E = \{x\}$ and let $\epsilon = \frac{1}{2} > 0$. The only measurable set containing E is $A = X$, and $\mu^*(A) = 2$ while $\mu^*(E) = 1$. So it's not true that

$$\mu^*(A) \leq \mu^*(E) + \epsilon,$$

and hence the outer measure is not regular.