Math 6320 Homework #10 Solution

11, 21a. *Show that, if $f$ is integrable, then the set $\{x : f(x) \neq 0\}$ is of $\sigma$-finite measure.*

Let $A = \{x : f(x) > 0\}$ and $B = \{x : f(x) < 0\}$. For each $n$ let $A_n = \{x : f(x) \geq \frac{1}{n}\}$ and let $B_n = \{x : f(x) \leq -\frac{1}{n}$.

Then $A_n \subset A$ and for every $n$,

$$\int_X f^+ \geq \int_A f^+ \geq \int_{A_n} f^+ \geq \int_{A_n} \frac{1}{n} = \frac{1}{n}\mu(A_n).$$

Thus

$$\mu(A_n) \leq n \int_X f^+ < \infty,$$

so in particular $A_n$ is of finite measure. Clearly $A = \cup_{n=1}^{\infty} A_n$ so that $A$ is $\sigma$-finite.

The same trick works for $B$ and $B_n$. Hence $A \cup B$ is the countable union of the $A_n \cup B_n$, and so is $\sigma$-finite.

11, 22a. *Let $(X, \mathcal{B}, \mu)$ be a measure space and $g$ a nonnegative measurable function on $X$. Set $\nu E = \int_E g\, d\mu$. Show that $\nu$ is a measure on $\mathcal{B}$.*

By definition of $\int_E g\, d\mu = \sup_{\varphi \leq g} \int_E \varphi$, with

$$\int_E \sum_{i=1}^{n} c_i \xi_{E_i}\, d\mu = \sum_{i=1}^{n} c_i \mu(E_i \cap E),$$

we see that $\int_\emptyset g\, d\mu = 0$. So $\nu\emptyset = 0$.

Now for countable additivity: let $A_i$ be a sequence of disjoint measurable sets, and let $A = \cup_{i=1}^{\infty} A_i$.

The easy proof is to notice that $\nu E = \int_E g\, d\mu = \int_X g\xi_E\, d\mu$ for any set $E \in \mathcal{B}$. By the monotone convergence theorem (or actually Corollary 14), we have

$$g\xi_A = \sum_{i=1}^{\infty} g\xi_{A_i},$$

so that

$$\nu(A) = \int_A g\,\mu = \int_X g\xi_A\,mu$$
$$= \int_X \sum_{i=1}^{\infty} g\xi_{A_i}\,\mu = \sum_{i=1}^{\infty} \int_X g\xi_{A_i}\,\mu = \sum_{i=1}^{\infty} \nu(A_i).$$

11, 25. *Give an example of a decreasing sequence $\langle \mu_n \rangle$ of measures on a measurable space such that the set function $\mu$ defined by $\mu E = \lim \mu_n E$ is not a measure.*

Let $m$ be Lebesgue measure on $\mathbb{R}$ and let $\mu_n = \frac{1}{n}m$. Then $\mu_n\mathbb{R} = \infty$ for every $n$, and thus $\mu\mathbb{R} = \infty$. On the other hand, $\mu_n[-m, m] = \frac{2m}{n}$ for every fixed $m$, so that $\mu[-m, m] = 0$ for every fixed $m$.

Thus $\mu\mathbb{R} \neq \lim_{m\to\infty} \mu[-m, m]$, so $\mu$ cannot be a measure.

11, 27b. *Show that the Hahn decomposition is unique except for null sets.*

Suppose $X = A \cup B$ and $X = C \cup D$ with $\nu$ positive on $A$ and $C$, $\nu$ negative on $B$ and $D$, and $A \cap B = C \cap D = \emptyset$.

Then

$$X = (A\cup B)\cap X = (A\cup B)\cap(C\cup D) = (A\cap(C\cup D))\cup(B\cap(C\cup D))$$
$$= (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D).$$

$\nu$ is positive on $A$ and negative on $D$, and since $A \cap D$ is a subset of both, we must have $\nu(A \cap D) = 0$. Similarly $\nu(B \cap C) = 0$. So letting $E = (A \cap D) \cup (B \cap C)$, we have $\nu(E) = 0$ and

$$X = (A \cap C) \cup (B \cap D) \cup E.$$

These sets are all disjoint. Thus since $B\cap D \subset B$, we know $\tilde{B} \subset \tilde{B}\cup \tilde{D}$, and $A = \tilde{B}$ implies

$$A \subset \tilde{B} \cup \tilde{D} = (A \cap C) \cup E.$$

Obviously we also have $C \subset (A \cap C) \cup E$. Thus

$$A\backslash C = A\cap \tilde{C} \subset \tilde{C}\cap\big((A\cap C)\cup E\big) = \big(\tilde{C}\cap(A\cap C)\big)\cup\big(\tilde{C}\cap E\big) = \tilde{C}\cap E.$$

Thus $A\backslash C$ is a set of measure zero. Similarly $C\backslash A$ is a set of measure zero. The same is true for $B$ and $D$.

11, 28. *Show that there is only one pair of mutually singular measures $\nu^+$ and $\nu^-$ such that $\nu = \nu^+ - \nu^-$. [Hint: Show that any such pair determines a Hahn decomposition and apply the results of Problem 27b.]*

Suppose $\mu^+$ and $\mu^-$ is another pair of mutually singular measures and $\nu = \mu^+ - \mu^-$. Let $A$ and $B$ be the disjoint sets with $X = A \cup B$ and $\nu^+(B) = 0$ and $\nu^-(A) = 0$, and let $C$ and $D$ be the disjoint sets with $X = C \cup D$ and $\mu^+(D) = 0$ and $\mu^-(C) = 0$.

Then for any measurable set $E \subset A$, we have $0 \leq \nu^-(E) \leq \nu^-(A) = 0$, so that $\nu(E) = \nu^+(E) - \nu^-(E) = \nu^+(E) \geq 0$. So $A$ is a positive set. Similarly $B$ is a negative set, $C$ is a positive set, and $D$ is a negative set.

By Problem 27b, $(A \cup C)\backslash(A \cap C)$ is a set of measure zero, as is $(B \cup D)\backslash(B \cap D)$, so that if $E$ is any set at all, then

$$\nu^+(E) = \nu^+(E\cap A) + \nu^+(E\cap B) = \nu^+(E\cap A) = \nu(E\cap A) - \nu^-(E\cap A)$$
$$= \nu(E \cap A) = \nu(E \cap A \cap C) + \nu(E \cap A \cap \tilde{C}) = \nu(E \cap A \cap C),$$

since $\nu(A \cap \tilde{C}) = 0$. Similarly $\mu^+(E) = \nu(E \cap A \cap C)$. Thus $\mu^+ = \nu^+$ on every set.

Similarly $\mu^- = \nu^-$ on every set.

11, 33a. *Show that the Radon-Nikodym Theorem for a finite measure $\mu$ implies the theorem for a $\sigma$-finite measure $\mu$. [Hint: Decompose $X$ into a countable union of sets $X_i$ of finite $\mu$-measure and apply the Radon-Nikodym Theorem to each $X_i$ to obtain $f$. Show $f$ to have the required properties.]*

As suggested, write $X = \cup_{n=1}^{\infty} X_n$ where $\mu(X_n) < \infty$. We can assume without loss of generality that the $X_n$ are disjoint, by taking

$$Y_n = X_n \cap \left( \widetilde{\bigcup_{k=1}^{n-1} X_k} \right).$$

For each $n$, we know there is a function $f_n$ such that whenever $E \in \mathcal{B}$ and $E \subset X_n$, we have

$$\nu(E) = \int_E f_n \, d\mu.$$

Define $f = \sum_{n=1}^{\infty} f_n \xi_{X_n}$. For any set $E$, let $E_n = E \cap X_n$. Then $E = \cup_{n=1}^{\infty} E_n$ is a disjoint union, and since $\nu$ is a signed measure we have

$$\begin{aligned} \nu(E) &= \sum_{n=1}^{\infty} \nu(E_n) \\ &= \sum_{n=1}^{\infty} \int_{E_n} f_n \, d\mu \\ &= \sum_{n=1}^{\infty} \int_X f_n \xi_{E_n} \, d\mu. \end{aligned}$$

We want to say

$$\sum_{n=1}^{\infty} \int_X f_n \xi_{E_n} \, d\mu = \int_X \sum_{n=1}^{\infty} f_n \xi_{E_n} \, d\mu,$$

which is true by the monotone convergence theorem since every $f_n \xi_{E_n}$ is nonnegative.

Then

$$\nu(E) = \int_X \sum_{n=1}^{\infty} f_n \xi_{E_n} = \int_X \sum_{n=1}^{\infty} f_n \xi_{X_n} \xi_E \, d\mu = \int_X \xi_E f \, d\mu = \int_E f \, d\mu.$$

This is true for every $E \in \mathcal{B}$, so $f$ is the function we want.

11, 39. *Use the following example to show that the hypothesis in the Radon-Nikodym Theorem that $\mu$ is $\sigma$-finite cannot be omitted. Let $X = [0, 1]$, $\mathcal{B}$ the class of Lebesgue measurable subsets of $[0, 1]$, and take $\nu$ to be Lebesgue measure and $\mu$ to be the counting measure on $\mathcal{B}$. Then $\nu$ is finite and absolutely continuous with respect to $\mu$, but there is no function $f$ such that $\nu E = \int_E f \, d\mu$ for all $E \in \mathcal{B}$. At what point does the proof of Theorem 23 break down for this example?*

We just have the check the things that are claimed. Certainly $\nu(X) = 1$ so $\nu$ is finite. Next, if $\mu(E) = 0$, then $E$ must be empty, so $\nu(E) = 0$ as well. Thus $\nu$ is absolutely continuous with respect to $\nu$.

Let $f$ be a nonnegative measurable function. For any $x$, let $E = \{x\}$. Then $E$ is Lebesgue measurable and $\nu(E) = 0$, while

$$\int_E f \, d\mu = f(x).$$

So if $\int_E f \, d\mu = \nu(E)$ for every singleton set, then $f(x) = 0$ for all $x$. Hence $\int_X f \, d\mu = 0$, but $\nu(X) = 1$.

If $E = \{x\}$ for any $x$ then for any positive rational $\alpha$ we have

$$\nu(E) - \alpha\mu(E) = 0 - \alpha < 0.$$

Hence no nonempty set is positive for $\nu - \alpha\mu$, and the Hahn decomposition is unique: it's always $A_\alpha = \emptyset$ and $B_\alpha = X$ for $\alpha > 0$. However we set $B_0 = \emptyset$ and $A_0 = X$. Thus we have $f \geq 0$ everywhere on $A_0 = X$, but $f \leq \alpha$ everywhere on $B_\alpha = X$ for $\alpha > 0$. Hence $f = 0$.

We have $B_\beta \backslash B_\alpha = B_\beta \cap \widetilde{B_\alpha} = X \cap \emptyset = \emptyset$ for every $\alpha, \beta > 0$, so that each $E_k$ is the empty set. But also $E_\infty = E \backslash X = \emptyset$.

So it is not true that $E = E_\infty \cup \bigcup_{k=0}^{\infty} E_k$, and in fact this is where the proof breaks down.