Math 6320 Homework #1 Solution

7, 3a We want to prove that the relation $x \equiv y \Leftrightarrow \rho(x, y) = 0$ is an equivalence, if $\rho$ is a pseudometric. Clearly $x \equiv x$ and $x \equiv y \Leftrightarrow y \equiv x$, so we have reflexivity and symmetry. For transitivity, suppose $x \equiv y$ and $y \equiv z$. Then

$$0 \leq \rho(x, z) \leq \rho(x, y) + \rho(y, z) = 0 + 0 = 0,$$

so $\rho(x, z) = 0$ and $x \equiv z$.

Now we want to prove $\rho$ depends only on the equivalence classes, i.e., if $x \equiv u$ and $y \equiv v$, then $\rho(x, y) = \rho(u, v)$. To do this, note that by the triangle inequality,

$$\rho(x, y) \leq \rho(x, u) + \rho(u, v) + \rho(v, y) = 0 + \rho(u, v) + 0 = \rho(u, v).$$

Similarly $\rho(u, v) \leq \rho(x, y)$.

Finally we want to check that $\rho$ is a metric on the equivalence classes. Nonnegativity, symmetry, and the triangle inequality all come from the pseudometric properties, and we just need to check positivity: if $[x] \neq [u]$ then $\rho([x], [u]) > 0$. Let $x$ and $u$ be any representatives of $[x]$ and $[u]$; then $\rho([x], [u]) = \rho(x, u)$. We know either $\rho(x, u) > 0$ or $\rho(x, u) = 0$; in the first case we are done; in the second case, $[x] = [u]$, as desired.

7, 3b We want to prove that the relation $x \equiv y \Leftrightarrow \rho(x, y) < \infty$ is an equivalence if $\rho$ is a metric with values in the extended reals. For reflexivity: $\rho(x, x) = 0 < \infty$ so $x \equiv x$. For symmetry:

$$x \equiv y \Rightarrow \rho(x, y) < \infty \Rightarrow \rho(y, x) < \infty \Rightarrow y \equiv x.$$

For transitivity: if $x \equiv y$ and $y \equiv z$, then $\rho(x, y) < \infty$ and $\rho(y, z) < \infty$, so

$$\rho(x, y) \leq \rho(x, y) + \rho(y, z) < \infty.$$

Now we want to prove that the equivalence class $[x]$ is open for any $x$. Clearly if $y \in [x]$ then for *every* $\delta > 0$, we have $B_\delta(y) \subset [x]$ since $\rho(y, z) < \delta$ implies $\rho(x, z) \leq \rho(x, y) + \delta < \infty$.

Finally we want to prove that $[x]$ is closed for any $x$. If $y$ is a closure point of $[x]$ then for every $\delta$ there is a $z \in [x]$ such that $\rho(y, z) < \delta$, and hence $\rho(x, y) \leq \rho(x, z) + \rho(z, y) < \rho(x, z) + \delta < \infty$. So $y \in [x]$.

7, 4a Prove $C$ is a closed subset of $L^\infty$. We identify $C$ with $[C]$, the set of functions $f$ on $[0,1]$ which agree with a continuous function $g$ on $[0,1]$ except on a set of measure zero. We have by definition that for any $f$ and $g$ in $L^\infty$,

$$\rho_{L^\infty}(f,g) = \inf\Big\{M \mid m\{t \,\big|\, |f(t)-g(t)| > M\} = 0\Big\}$$

Our strategy will essentially be the following: we first reduce the set we are thinking of from *equivalence classes* of continuous functions to *actual* continuous functions; then show that we can reduce the $L^\infty$ norm to the $C^0$ (supremum) norm; and finally use the fact that continuous functions are complete in the $C^0$ norm.

So suppose $f$ is a function which is a point of closure of $C([0,1])$, i.e., for every $\delta > 0$ there is a continuous $g$ such that $\rho_{L^\infty}(f,g), \delta$. (Here we are thinking of $f$ as a representative of some equivalence class, while $g$ is the unique continuous function in its equivalence class.) For each positive integer $n$, choose a continuous $g_n$ such that $\rho_{L^\infty}(g_n, f) < \frac{1}{n}$. Then

$$\rho_{L^\infty}(g_n, g_m) \leq \rho_{L^\infty}(g_n, f) + \rho_{L^\infty}(g_m, f) < \frac{1}{m} + \frac{1}{n},$$

and we can easily see that $< g_n >$ is a Cauchy sequence in $L^\infty$.

Now we prove that for any two continuous functions $g_m$ and $g_n$,

$$\rho_{L^\infty}(g_m, g_n) = \rho_{C^0}(g_m, g_n).$$

To do this, note that if $M > 0$ and $E_M = \{t \,|\, |f_n(t) - f_m(t)| > M\}$, then $\rho_{L^\infty} = \inf\{M \,|\, m(E_M) = 0\}$. Hence if we can prove that $m(E_M) = 0$ iff $E_M =$ for all $M$, then we will have $\rho_{L^\infty} = \rho_{C^0}$. So suppose $E_M$ is nonempty, and let $t_o$ be a point such that $|f_n(t_o) - f_m(t_o)| > M$. Define $\varepsilon > 0$ by $|f_n(t_o) - f_m(t_o)| - M = 2\varepsilon$. Since $f_n$ and $f_m$ are continuous at $t_o$, there is a $\delta > 0$ such that $|t - t_o| < \delta$ implies both $|f_n(t) - f_n(t_o)| < \varepsilon$ *and* $|f_m(t) - f_m(t_o)| < \varepsilon$. Hence $|t - t_o| < \delta$ implies

$$\begin{aligned} |f_n(t_o) - f_m(t_o)| &\leq |f_n(t_o) - f_n(t)| + |f_n(t) - f_m(t)| + |f_m(t) - f_m(t_o)| \\ &< 2\varepsilon + |f_n(t) - f_m(t)|, \end{aligned}$$

from which we conclude that $|t - t_o| < \delta$ implies $|f_n(t) - f_m(t)| > |f_n(t_o) - f_m(t_o)| - 2\varepsilon = M$. In particular $E_M$ contains $(t_o - \delta, t_o + \delta)$.

We have thus proved that if $E_M$ is nonempty, then it has positive measure; in other words, if $E_M$ has measure zero, it must be empty.

Since $\rho_{L^\infty}$ and $\rho_{C^0}$ agree on continuous functions, the sequence $< g_n >$ is a Cauchy sequence in $C^0$. Since $C^0$ is complete, $g_n$ converges to some continuous function $g$ in the $C^0$ norm, and hence also in the $L^\infty$ norm. But $g_n$ also converges to $f$, the point of closure of $[C]$ in the $L^\infty$ norm. Hence $f$ is equivalent to a continuous function $g$, which is what we wanted to prove.

- Which of the spaces $\mathbb{R}^n$, $C$, $L^\infty$, $L^1$ are separable?

  $\mathbb{R}^n$ is separable since $\mathbb{Q}^n$ is a dense subset, and is countable as a finite product of countable sets.

  $C([0,1])$ is separable since we can approximate any continuous function uniformly in the supremum norm by polynomials (the Weierstrass theorem). We can also approximate polynomials uniformly on $[0,1]$ by polynomials with rational coefficients. The set of polynomials with rational coefficients is a countable union of countable sets, hence also countable.

  $L^1$ is separable since by Proposition 6.4.8, any element can be approximated in $L^1$ by a step function, and we can approximate a step function in $L^1$ by one with rational endpoints and rational coefficients. This is a countable union of countable sets.

  $L^\infty$ is *not* separable. In fact we cannot even approximate simple step functions. For each $x \in (0,1)$ let $u_x$ be the Heaviside step function

$$u_x(t) = \begin{cases} 0 & t < x, \\ 1 & t \geq x. \end{cases}$$

  Then if $x < y$ and $M < 1$, we have

$$\{t \,|\, |u_x(t) - u_y(t)| > M\} = \{t \,|\, |u_x(t) - u_y(t)| = 1\} = \{t \,|\, x \leq t < y\},$$

  and the measure of this set is $y - x > 0$. Hence $\rho_{L^\infty}(u_x, u_y) = 1$ whenever $x \neq y$. We therefore have uncountably many functions all of which are distance 1 away from all others. Any dense subset would have to have at least one element within distance $\frac{1}{2}$ of each of these functions, and in particular it would have to have at least as many elements as $(0,1)$, i.e., uncountably many.

7, 10a If the condition holds, then clearly the identity is continuous in both directions. Conversely if the identity map is continuous in both directions at $x$, then for every $\varepsilon$ there is a $\delta_1$ that works in one direction and a $\delta_2$ that works in the other. We choose $\delta = \min\{\delta_1, \delta_2\}$ and we're done.

7, 10b We obviously have

$$\rho(x,y) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2} \leq \sqrt{n} \max\{|x_k - y_k|\} = \sqrt{n}\rho^+(x,y)$$

while

$$\rho^+(x,y) = \max\{|x_k - y_k|\} \leq \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2} = \rho(x,y).$$

So we can choose $\delta = \frac{\varepsilon}{\sqrt{n}}$.

Similarly $\rho^*(x,y) \leq n\rho^+(x,y)$ and $\rho^+(x,y) \leq \rho^*(x,y)$, so $\delta = \frac{\varepsilon}{n}$ works.

Since $\rho$ and $\rho^*$ are both equivalent to $\rho^+$, it's easy to see that they are equivalent to each other.

7, 17a We want to prove that if $< x_n >$ and $< y_n >$ are Cauchy sequences in $X$, then $\rho(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$. We have

$$\rho(x_n, y_n) \leq \rho(x_n, x_m) + \rho(x_m, y_m) + \rho(y_m, y_n)$$

and

$$\rho(x_m, y_m) \leq \rho(x_m, x_n) + \rho(x_n, y_n) + \rho(y_n, y_m),$$

so we conclude that

$$|\rho(x_m, y_m) - \rho(x_n, y_n)| \leq \rho(x_m, x_n) + \rho(y_m, y_n).$$

Let $\varepsilon > 0$ and choose $N$ so that $m, n \geq N$ implies both $\rho(x_m, x_n) < \frac{\varepsilon}{2}$ and $\rho(y_m, y_n) < \frac{\varepsilon}{2}$. Then $n, m \geq N$ implies

$$|\rho(x_m, y_m) - \rho(x_n, y_n)| < \varepsilon.$$

Hence $< \rho(x_n, y_n) >$ is a Cauchy sequence in $\mathbb{R}$, so it converges in $\mathbb{R}$.

7, 17b Nonnegativity and symmetry are obvious; we just need to prove the triangle inequality. For any sequences $< x_n >$, $< y_n >$ and $< z_n >$, we have

$$\rho(x_n, y_n) \leq \rho(x_n, z_n) + \rho(z_n, y_n)$$

for every $n$. In particular if these are Cauchy sequences and the limits exist, then

$$\lim_{n\to\infty} \rho(x_n, y_n) \leq \lim_{n\to\infty} \rho(x_n, z_n) + \lim_{n\to\infty} \rho(z_n, y_n).$$

7, 17c That $X^*$ is a metric space comes directly from problem 3. For the isometric embedding, we just take $f\colon X \to X^*$ given by $f(x) = [< x, x, x, \ldots >]$. Then

$$\rho(f(x), f(y)) = \lim_{n\to\infty} \rho(x, y) = \rho(x, y).$$

7, 17d We want to prove $X^*$ is complete, so let $\mathbf{x}_n$ be a sequence of equivalence classes of Cauchy sequences, which forms a Cauchy sequence in the metric $\rho^*$. That is, for each fixed $n$, $< x_{nk} >$ is a Cauchy sequence in $\rho$, so that for each $n$ and for any $\varepsilon > 0$ there is an $N$ such that $j, k \geq N$ implies $\rho(x_{nj}, x_{nk}) < \varepsilon$. Also for any $\varepsilon > 0$ there is an $N$ such that $n, m \geq N$ implies $\rho^*(\mathbf{x}_n, \mathbf{x}_m) = \lim_{k\to\infty} \rho(x_{nk}, x_{mk}) < \varepsilon$.

Since we can choose *any* representative for a Cauchy sequence, we may as well choose one that converges quickly. So given any $\mathbf{x}_n$, choose a subsequence satisfying $\rho(x_{nk}, x_{n,k+1}) \leq \frac{1}{2^{k+1}}$. Then it is easy to see using the triangle inequality and geometric series that

$$\rho(x_{nj}, x_{nk}) \leq \frac{1}{2^{\max\{j,k\}}}.$$

Now let's prove that the sequence of diagonals $< x_{nn} >$ is itself a Cauchy sequence. We have for any $n$ and $m$, and any $k > \max\{m, n\}$, that

$$\begin{aligned}\rho(x_{nn}, x_{mm}) &\leq \rho(x_{nn}, x_{nk}) + \rho(x_{nk}, x_{mk}) + \rho(x_{mk}, x_{mm}) \\ &\leq \frac{1}{2^n} + \rho(x_{nk}, x_{mk}) + \frac{1}{2^m}.\end{aligned}$$

Now take the limit as $k \to \infty$, and we obtain for every $n$ and $m$ that

$$\rho(x_{nn}, x_{mm}) \leq \frac{1}{2^n} + \frac{1}{2^m} + \rho^*(\mathbf{x}_n, \mathbf{x}_m).$$

Since $< \mathbf{x}_n >$ is a Cauchy sequence in $\rho^*$, there is an $N_1$ such that $n, m \geq N_1$ implies $\rho^*(\mathbf{x}_n, \mathbf{x}_m) < \frac{\varepsilon}{3}$. Clearly there is an $N_2$ such that $n \geq N_2$ implies $\frac{1}{2^n} < \frac{\varepsilon}{3}$. Hence $n, m \geq N = \max\{N_1, N_2\}$ is enough to imply $\rho(x_{nn}, x_{mm}) < \varepsilon$. So indeed $< x_{nn} >$ is a Cauchy sequence.

Finally we want to prove that $\lim_{n\to\infty} \mathbf{x}_n =< x_{kk} >$ in $\rho^*$, i.e., that

$$\lim_{n\to\infty} \lim_{k\to\infty} \rho(x_{nk}, x_{kk}) = 0.$$

Assuming $j > k > n$, we have

$$\rho(x_{nk}, x_{kk}) \leq \rho(x_{nk}, x_{nj}) + \rho(x_{nj}, x_{kj}) + \rho(x_{kj}, x_{kk}) \leq \frac{1}{2^{k-1}} + \rho(x_{nj}, x_{kj}),$$

and this being true for every $j$ yields

$$\rho(x_{nk}, x_{kk}) \leq \frac{1}{2^{k-1}} + \rho^*(\mathbf{x}_n, \mathbf{x}_k).$$

Since $< \mathbf{x}_n >$ is Cauchy, we can choose $N$ so large that $k, n \geq N$ implies both $\frac{1}{2^{k-1}} < \frac{\varepsilon}{2}$ and $\rho^*(\mathbf{x}_n, \mathbf{x}_k) < \frac{\varepsilon}{2}$. So for $n, k \geq N$ we have

$$\rho(x_{nk}, x_{kk}) \leq \varepsilon.$$

Hence

$$\lim_{k\to\infty} \rho(x_{nk}, x_{kk}) \leq \varepsilon$$

as long as $n \geq N$. Thus finally we have

$$\lim_{n\to\infty} \lim_{k\to\infty} \rho(x_{nk}, x_{kk}) = 0,$$

which shows that the original sequence of Cauchy sequences converges to the diagonal Cauchy sequence.