Math 6320 Midterm Solutions

1. (a) Let X_0 be the space of functions f on $[0,1]$ such that f' exists almost everywhere, $f(0)=0$, and $f' \in L^2$. Define

$$\|f\|_{W^2} = \sqrt{\int_0^1 f'(x)^2\,dx}.$$

Is this a norm?

Answer: No. Although it inherits the Minkowski inequality from the L^2 norm on which it's based, and symmetry and nonnegativity are obvious, it is *not* nondegenerate.

As a simple example, if

$$f(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2}, \\ 1 & \frac{1}{2} \le x \le 1, \end{cases}$$

then $f'(x) = 0$ for almost all x. In particular f' is in L^2 and

$$\|f\|_{W^2} = \|f'\|_{L^2} = 0.$$

Yet f is not equal to zero, even up to measure-zero equivalence class.

A more complicated example is the Cantor ternary function, which is continuous and surjective onto $[0,1]$. Its derivative exists (and is zero) everywhere outside the Cantor set; in particular it is differentiable almost everywhere. Yet of course it's not equal to the zero function, even up to equivalence.

(b) Let $M > 0$ and let $D_M = \{f \in X_0 \mid \|f\|_{W^2} \le M\}$. Does $D_M \cap C[0,1]$ have compact closure in $C[0,1]$?

Answer: Almost, but not quite. The temptation is to use Cauchy-

Schwarz and get

$$\begin{aligned}
|f(x)-f(y)| &\leq \int_y^x |f'(t)|\,dt \\
&\leq \sqrt{\left(\int_y^x dt\right)\left(\int_y^x |f'(t)|^2\,dt\right)} \\
&\leq \sqrt{|x-y|}\sqrt{\int_0^1 f'(t)^2\,dt} \\
&\leq M\sqrt{|x-y|}.
\end{aligned}$$

However this is only valid if the fundamental theorem of calculus is valid for the function, and that's *not* always the case, even if the function is differentiable almost everywhere.

The basic counterexample is again the Cantor ternary function, which is differentiable almost everywhere, so that the Lebesgue integral of its derivative exists everywhere and is equal to the zero function. The scalar multiples of the Cantor function form a subspace contained in $D_M \cap C[0,1]$, which is therefore not even bounded. So it certainly can't be compact.

2. Let f be a C^∞ function such that for every $x \in [0,1]$ there is an integer k such that $f^{(k)}(x) = 0$. Prove that there is an interval $(a,b) \subset [0,1]$ such that f is a polynomial on (a,b).

   Answer: The only tool we have for turning pointwise facts into uniform facts is the Baire category theorem. So this must have something to do with nowhere-denseness.

   For any integer $k \geq 0$ let $E_k = \{x : f^{(k)}(x) = 0\}$. Since every $x \in [0,1]$ is in some E_k by assumption, we have

   $$\bigcup_{k=0}^{\infty} E_k = [0,1].$$

   Now $[0,1]$ is complete, so it can't be first category by Corollary 7.28; in particular at least one E_k is not nowhere dense. Now since f is C^∞, we know each $f^{(k)}$ is continuous, so each E_k is closed. Hence at least one E_k has nonempty interior.

So there is an open interval $(a, b) \subset E_k$ for some k. On this interval $f^{(k)} = 0$, so f must be a polynomial of degree at most $(k-1)$ on this interval.

In fact one can prove that f must be a polynomial on all of $[0, 1]$, but that's a little harder.

3. If X is a totally bounded metric space and $K(X)$ is the set of all closed subsets of X, then the distance on $K(X)$ is defined as

$$R(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} \rho(x, y), \sup_{y\in B} \inf_{x\in A} \rho(x, y)\}.$$

   Prove R is a metric. Prove that R is totally bounded.

   Answer: This is not too hard as long as one keeps the definitions straight. For example using the correct definition from the beginning helps a lot.

   The fact that $R(A, B)$ is finite for every closed sets A and B follows from total boundedness of X, which in particular implies boundedness of X. That $R(A, B)$ is nonnegative and symmetric is obvious from the definitions. The things to prove are positive-definiteness and the triangle inequality.

   To prove positive-definiteness, suppose $R(A, B) = 0$. Then $\sup_{x\in A} \inf_{y\in B} \rho(x, y) = 0$. In particular for every $x \in A$ we have $\inf_{y\in B} \rho(x, y) = 0$. Hence every $x \in A$ is a limit of points in B, and since B is closed, every $x \in A$ must be contained in B. So $A \subset B$. Using $\sup_{y\in B} \inf_{x\in A} \rho(x, y) = 0$, we get $B \subset A$. So $A = B$.

   Now to prove the triangle inequality, let C be any closed set. If $x \in A$, $y \in B$, and $z \in C$, then

$$\rho(x, y) \le \rho(x, z) + \rho(z, y)$$

   by the usual triangle inequality. Now take the infimum first over y, then over z:

$$\inf_{y\in B} \rho(x, y) \le \inf_{z\in C} \rho(x, z) + \inf_{z\in C} \inf_{y\in B} \rho(z, y)$$

   for every x. Now

$$\inf_{z\in C} \inf_{y\in B} \rho(z, y) \le \sup_{z\in C} \inf_{y\in B} \rho(z, y) \le R(C, B).$$

So for every x we have

$$\inf_{y\in B}\rho(x,y)\leq\inf_{z\in C}\rho(x,z)+R(C,B).$$

Now take the supremum over all x, and we get

$$\sup_{x\in A}\inf_{y\in B}\rho(x,y)\leq\sup_{x\in A}\inf_{z\in C}\rho(x,z)+R(C,B)\leq R(A,C)+R(C,B).$$

By symmetry the same trick works the other way around, and we get

$$\sup_{y\in B}\inf_{x\in A}\rho(x,y)\leq R(B,C)+R(C,A).$$

So taking the maximum of both, we get

$$R(A,B)\leq R(A,C)+R(C,B).$$

Now finally we have to prove total boundedness of $K(X)$ under the metric R, using only total boundedness of X. Recall the definition of total boundedness of X: for every $\varepsilon>0$ there is a finite set $E=\{x_1,\ldots,x_n\}$ such that for every $y\in X$ there is an x_k such that $\rho(y,x_k)<\varepsilon$. So let $\varepsilon>0$; we want to find sets that can approximate any other set. Consider the set of all subsets of E, 2^E. This is a finite closed set.

Let A be any closed set. Let $Y=\{y\in X:\rho(y,A)<\varepsilon\}$. Let $B=Y\cap E$. Then $B\in 2^E$, and we want the distance from A to B. First for any $x\in A$ we have

$$\inf_{y\in B}\rho(x,y)<\varepsilon$$

so that

$$\sup_{x\in A}\inf_{y\in B}\rho(x,y)\leq\varepsilon.$$

Next for any fixed $y\in B$ we can find an $x\in A$ such that $\rho(x,y)<\varepsilon$ since $y\in Y$; thus

$$\inf_{x\in A}\rho(x,y)<\varepsilon.$$

Thus

$$\sup_{y\in B}\inf_{x\in A}\rho(x,y)\leq\varepsilon.$$

So we have $R(A,B)\leq\varepsilon$. In other words, every closed set A is within ε of some element of 2^E. So $K(X)$ is totally bounded.

4. Consider $(\ell^\infty)^*$ (the linear dual of the space of *bounded sequences, not sequences in* $[0,1]$*, oops again*) and $\beta(\mathbb{N})$ (the Stone-Cech compactification of $\mathbb{N}$) as sets of functions on ℓ^∞. What can you say about the elements in the intersection of these spaces?

   Answer: First recall that $\beta(\mathbb{N})$ is the closure of the image of $\mathbb{N}$ in $[-1,1]^{C(\mathbb{N},[-1,1])}$. Now $C(\mathbb{N},[-1,1])$ is the set of all sequences $\langle x_n \rangle$ with $-1 \leq x_n \leq 1$ for all n (since $\mathbb{N}$ is discrete, every real-valued function is continuous). So elements of $\beta(\mathbb{N})$ are functions on sequences in $[-1,1]$.

   Now ℓ^∞ is not quite the same as $C(\mathbb{N},[-1,1])$; the latter is the unit ball in ℓ^∞. A continuous linear functional F on ℓ^∞ is a map into $[-1,1]$ if and only if $\|F\|_{(\ell^\infty)^*} \leq 1$; otherwise there is at least one sequence $\mathbf{x}$ bounded by 1 with $|F(\mathbf{x})| > 1$. So if we have a bounded linear functional F with $\|F\|_{(\ell^\infty)^*} \leq 1$, its restriction to the unit ball in ℓ^∞ is an element of $[-1,1]^{C(\mathbb{N},[-1,1])}$. So it's at least a candidate to be an element of $\beta(\mathbb{N})$.

   Let us denote the evaluation function by E_k, so that

   $$E_k(\mathbf{x}) = E_k(\langle x_1, x_2, x_3, \ldots \rangle) = x_k$$

   for every sequence. Recall from the homework solutions that a function F on sequences in $[-1,1]$ is in $\beta(\mathbb{N})$ if and only if, for every finite collection of sequences $\mathbf{x}_1, \ldots, \mathbf{x}_n$, and for every $\varepsilon > 0$, there is a $k \in \mathbb{N}$ such that

   $$|F(\mathbf{x}_i) - E_k(\mathbf{x}_i)| < \varepsilon.$$

   The first step is to figure out what F does on the simplest elements of ℓ^∞, the "basis" elements e_j consisting of 1 in the j^{th} place and zeroes elsewhere. I claim first that $F(e_j) = 0$ or $F(e_j) = 1$ for every j. Suppose not; then let

   $$\varepsilon = \min\{|F(e_j)|, |F(e_j) - 1|\} > 0.$$

   Then there is a k such that

   $$|F(e_j) - \delta_{kj}| < \min\{|F(e_j)|, |F(e_j) - 1|\},$$

   which is impossible. So for any j, either $F(e_j) = 0$ or $F(e_j) = 1$.

I claim next that it's impossible to have $F(e_{j_1}) = 1$ and $F(e_{j_2}) = 1$ for $j_1 \neq j_2$. This is because, taking $\varepsilon = \frac{1}{2}$, there must be a single k such that

$$|1 - \delta_{j_1 k}| < \varepsilon \quad \text{and} \quad |1 - \delta_{j_2 k}| < \varepsilon,$$

which means $j_1 = j_2$.

So suppose $F(e_j) = 1$ for some j; I claim that $F = E_j$. Take any sequence $\mathbf{x}$; then for any $\varepsilon > 0$ there is a k such that both $|F(\mathbf{x}) - E_k(\mathbf{x})| < \varepsilon$ and also $|F(e_j) - E_k(e_j)| < \varepsilon$. If $\varepsilon \leq 1$ then this second inequality forces $k = j$. So we see $|F(\mathbf{x}) - E_j(\mathbf{x})| < \varepsilon$. This is true for any ε and any sequence $\mathbf{x}$, so that indeed $F = E_j$.

The only other case is when $F(e_j) = 0$ for every j. We are tempted to say this means $F \equiv 0$, but that's not immediately obvious. Notice the e_j are *not* dense in ℓ^∞. We can't approximate $\mathbf{1} = \langle 1, 1, 1, \ldots \rangle$ by such elements in the ℓ^∞ norm, for example. And in fact we must clearly have $F(\mathbf{1}) = 1$ since $E_k(\mathbf{1}) = 1$ for every k.

I claim that if $F(e_j) = 0$ for every j, then for any sequence $\mathbf{x}$, the number $F(\mathbf{x})$ is a limit of some subsequence of $\mathbf{x}$. Now let $\varepsilon > 0$ be any number, and let n be any positive integer. Then there is a k such that

$$|F(\mathbf{x}) - E_k(\mathbf{x})| < \varepsilon \quad \text{and} \quad |F(e_j) - E_k(e_j)| < \varepsilon \text{ for } 1 \leq j \leq n.$$

So $k \geq n$. Hence I have shown that for every $\varepsilon > 0$ and any positive integer n there is a $k \geq n$ such that

$$|F(\mathbf{x}) - x_k| < \varepsilon.$$

In other words, $F(\mathbf{x})$ is a limit point of the sequence, and there is a subsequence converging to F.

So if such an F exists, then it must be a limit functional. However it can't be a Banach limit, since the Banach limit of $\langle 1, -1, 1, -1, \ldots \rangle$ is always zero, yet zero is not a limit point of this sequence.

I don't know whether such an F can actually exist. One idea for trying to show one doesn't exist is to do the following. For every subset $J \subset \mathbb{N}$, construct a sequence $\mathbf{x}$ by

$$x_k = \begin{cases} 1 & k \in J \\ 0 & k \notin J \end{cases}.$$

There are certainly uncountably many such subsets J, and hence uncountably many such sequences. For every such $\mathbf{x}$, clearly either $F(\mathbf{x}) = 1$ or $F(\mathbf{x}) = 0$. Furthermore if $\mathbf{x}$ is the sequence corresponding to J and $\mathbf{y}$ is the sequence corresponding to the complement of J, then $\mathbf{x} + \mathbf{y} = \mathbf{1}$, so by linearity $F(\mathbf{x}) = 1 - F(\mathbf{y})$.

Let $\mathcal{S}$ be the set of all such subsets of $\mathbb{N}$ with $F(\mathbf{x}) = 1$ for the corresponding sequence. Then $J \in \mathcal{S}$ iff $\tilde{J} \notin \mathcal{S}$. If $\varepsilon = \frac{1}{2}$, then for any finite collection $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of sequences coming from sets in S, there is a k such that $x_{jk} = 1$ for $1 \leq j \leq n$. In other words, $\mathcal{S}$ has the finite intersection property. This seems impossible to me, though I couldn't see how to prove it.

Oh well. I thought this was the hardest problem on the exam, and I generally gave credit as long as you said something nontrivial about it (hence why the question was open-ended).

5. Is $\mathbb{R}^{\mathbb{N}}$ separable in the product topology? Is $\mathbb{R}^{[0,1]}$ separable in the product topology?

   Answer: Yes to both. We need to find a countable dense subset, which is equivalent to finding a countable set D such that every basis element contains an element of D. So we need to understand the basis elements.

   The basis elements of any product topology X^A are finite intersections of subbasis elements, which are of the form

$$\Pi_{a \neq b} X \times U_b.$$

   More explicitly, any basis element in $\mathbb{R}^{\mathbb{N}}$ is, for some $\{n_1, \cdots, n_m\}$, of the form

$$O = \{x \colon \mathbb{N} \to \mathbb{R} : x_{n_k} \in (a_k, b_k).$$

   So set $D = \{\langle q_1, q_2, \ldots, q_n, 0, 0, \ldots\rangle : n \in \mathbb{N}, q_k \in \mathbb{Q}\}$. Clearly for any open set O of the form shown, there is an element of D contained in it. Also for each n the product $\mathbb{Q}^n$ is countable, and D is a countable union of these sets. So D is also countable.

   If we think of $\mathbb{R}^{[0,1]}$ as the set of functions from $[0,1]$ to $\mathbb{R}$, then the basis elements are of the form

$$\{f \colon [0,1] \to \mathbb{R} : f(x_k) \in (a_k, b_k)\}$$

for some numbers $x_k, a_k, b_k \in [0,1]$ with $1 \leq k \leq n$. Now for each n and each set of rationals $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ such that $p_1 < q_1 < p_2 < q_2 < \cdots < p_n < q_n$ and each set of rationals $\{r_1, \ldots, r_n\}$, define a function

$$f(x) = \begin{cases} r_k & x \in (p_k, q_k) \text{ for some } k \\ 0 & \text{otherwise} \end{cases}.$$

Now for any fixed n there are less than $\mathbb{Q}^{3n}$ such functions, so countably many. Taking the union over all n, we still get a countable set. So indeed, $\mathbb{R}^{[0,1]}$ is separable.

6. Let X be an infinite-dimensional normed vector space, and let $Q = \{f \in X \mid \|f\| = 1\}$. Prove that the weak closure of Q is the closed unit ball $B = \{f \in X \,|\|f\| \leq 1\}$. (Hint: prove any weak open set containing zero must also contain a subspace.)

   Answer: We recall the weak topology on X. Its basis elements look like
   $$g_1^{-1}(a_1, b_1) \cap g_2^{-1}(a_2, b_2) \cap \cdots \cap g_n^{-1}(a_n, b_n),$$
   for numbers $a_k < b_k$ and bounded linear functionals g_k. Now such a weak-open set contains zero if and only if $a_k < 0 < b_k$ for every k, since $g_k(\theta) = 0$ for every k.

   To prove this open set contains a subspace, we just need to show that for every finite collection of linear functionals g_k there is a vector x such that $g_k(x) = 0$ for all k. Now the image of each g_k has dimension at most one, and the image of $\{g_1, \cdots, g_n\}$ has dimension at most n. But the dimension of X is larger than n, so the kernel of $\{g_1, \cdots, g_n\}$ must be nonempty.

   Let's prove that B is weak closed, or equivalently that $\widetilde{B}$ is weak open. So let $f \in \widetilde{B}$, i.e., $\|f\| > 1$. Define a linear functional on the space spanned by f so that $g(f) = \|f\|$. Then by the Hahn-Banach theorem we can extend g so that
   $$|g(x)| \leq \|x\|$$
   for all x. In particular if $x \in B$ then $|g(x)| \leq 1$. So if $r \in (1, \|f\|)$, then $g^{-1}(r, \infty)$ is a weak open set containing f and completely contained in $\widetilde{B}$. So $\widetilde{B}$ is weak open, and hence B is weak closed. So the weak closure of Q is contained in B.

Now let S be the weak closure of Q. Then $\widetilde{S}$ is weak open. Since $Q \subset S$, we know $\widetilde{S} \subset \widetilde{Q}$, i.e., every $x \in \widetilde{S}$ has $\|x\| \neq 1$. Also since $S \subset B$ we know that $\widetilde{B} \subset \widetilde{S}$, i.e., every $x \in X$ with $\|x\| > 1$ is in $\widetilde{S}$. If there is a $y \in B \cap \widetilde{S}$, then $\|y\| < 1$; furthermore $\widetilde{S} - y$ is an open set containing the zero vector, so it contains a subspace T by the reasoning above. So there is a nonzero vector w such that $y + \lambda w \in \widetilde{S}$ for every $\lambda \in \mathbb{R}$.

Now $\|y + \lambda w\| \geq \|\lambda w\| - \|y\| = |\lambda|\|w\| - \|y\|$. So choosing $\lambda = \frac{\|y\|}{\|w\|}$, we see that we can make elements of $\widetilde{S}$ as large as we want in norm. Now the function $\lambda \mapsto \|y + \lambda w\|$ is continuous, and since it's less than one when $\lambda = 0$ and greater than one for some values of λ, it must attain the value one for some λ. But this is a contradiction, because no element of $\widetilde{S}$ can have norm one. So there is no $y \in B \cap \widetilde{S}$, and we must have $B \subset S$.

So the weak closure of Q is B.