Math 6320 Final

1. *Let* $(\mathbb{R}^2, \mathcal{S}, \lambda)$ *be the product measure in the sense of Royden, where* $X = Y = \mathbb{R}$*,* $\mu = \nu$ *is Lebesgue measure, and* $\mathcal{A} = \mathcal{B}$ *is the* σ*-algebra of Lebesgue measurable sets.*

   Prove that any open set in $\mathbb{R}^2$ *is measurable.*

   The products of open intervals form a subbasis for the product topology in $\mathbb{R}^2$, by definition of the product topology in general. Certainly open intervals are measurable in $\mathbb{R}$, so these are measurable sets in the product measure.

   Hence finite intersections of products of open intervals are also measurable, and so there is a basis of measurable sets.

   Furthermore $\mathbb{R}^2$ is separable (a countable dense subset is the pairs of rationals), so it's second countable. Thus we only need countably many finite intersections of products of open intervals to form a basis for $\mathbb{R}^2$.

   So any open set in $\mathbb{R}^2$ is a countable union of measurable sets, and thus is measurable.

   More generally, any product measure for which the measures on each factor are Borel and the product topology is second countable will be Borel.

2. *Let* ℓ^2 *denote the Hilbert space*

   $$\ell^2 = \left\{ (x_1, x_2, \ldots) \,\middle|\, x_k \in \mathbb{R}\, \forall k \in \mathbb{N},\ \sum_{k=1}^{\infty} x_k^2 < \infty \right\},$$

   with inner product given by

   $$\langle (x_1, x_2, \ldots), (y_1, y_2, \ldots) \rangle = \sum_{k=1}^{\infty} x_k y_k.$$

   Suppose μ *is a Borel measure on* ℓ^2 *which is invariant under isometries; in other words, if* $\Phi\colon \ell^2 \to \ell^2$ *is such that* $\langle \Phi(x-y), \Phi(x-y) \rangle = \langle x-y, x-y \rangle$ *for all* $x, y \in \ell^2$*, then for any measurable set* E*, the set* $\Phi(E)$ *is also measurable and* $\mu(\Phi(E)) = \mu(E)$*.*

Prove that for any open set U, either $\mu(U) = 0$ or $\mu(U) = \infty$. (Hint: isometries include translations, rotations, and the shift map $(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$.)

There are several ways to do this. An easy one is to suppose there is a ball B_0 of finite nonzero measure. Translate it to be centered at the origin (which does not change the measure). Apply the shift map to it to get B_1. Then $B_1 \subset B_0$ strictly. However $\mu B_1 = \mu B_0$.

By itself this is no big deal, but if $B_{k+1} = \Phi(B_k)$ for every k, then we get a nested sequence $B_{k+1} \subset B_k$ of measurable sets such that $\mu(B_k) = \mu(B_0)$ for every k.

However the intersection of all B_k is the singleton set consisting of the zero vector. But since B_0 has finite measure, we know by Proposition 11.2 that

$$\mu(\{0\}) = \mu\left(\bigcap_{i=0}^{\infty} B_i\right) = \lim_{i\to\infty} \mu(B_i) = \mu(B_0).$$

So any singleton point has finite and nonzero measure, by translation invariance. Yet there are obviously infinitely many distinct points x_i contained in B_0, and by countable additivity we see that $\mu(B_0) \geq \sum_{i=1}^{\infty} \mu(x_i) = \infty$. Contradiction.

Another way to prove this is to show that for any ball, there are infinitely many disjoint balls of a smaller radius contained in the large ball to get the same kind of contradiction.

3. *Let X be a two-point space $X = \{x, y\}$ with $\mathcal{B} = 2^X$. Show that there is a measure μ on X such that $(L^1(X))^* \neq L^\infty(X)$ (work everything out explicitly).*

   A big hint is the Riesz representation theorem, which says this is true if X is σ-finite. So clearly X cannot be finite to get the counterexample, and furthermore at least one of the singletons must have infinite measure (if both had finite measure, X would be σ-finite).

   So consider the measure $\mu(\emptyset) = 0$, $\mu\{x\} = 1$, $\mu\{y\} = \infty$, and $\mu(X) = \infty$. Check that this is actually countably additive: the only thing to worry about is $\mu\{x\} + \mu\{y\} = \mu(X)$, which is certainly true. So this actually is a measure with the correct measurable sets.

Since every set is measurable, every function is measurable, so to find the functions in L^1 we just have to understand integration. This is easy; since there are only finitely many points, every function is a simple function, and

$$\int_X f\,d\mu = f(x)\mu\{x\} + f(y)\mu\{y\}.$$

Now since $\mu\{y\} = \infty$, the function f is integrable iff $f(y) = 0$. In that case

$$\int_X |f|\,d\mu = |f(x)|\mu\{x\} = |f(x)|.$$

So $L^1(X)$ is a one-dimensional vector space isomorphic to $\mathbb{R}$ with its usual absolute-value norm. That means $(L^1(X))^*$ is also one-dimensional by general principles of vector spaces. Explicitly, a functional is completely determined by what it does to the vector $f = (f(x), 0)$, and by linearity all it can do is multiply $f(x)$ by some number. So if F is any functional on L^1, then there is a $\lambda \in \mathbb{R}$ such that $F(f) = \lambda f(x)$ for every $f \in L^1$. Furthermore the norm of F is $\|F\|_{(L^1)^*} = |\lambda|$, so $(L^1)^*$ is also isomorphic to $\mathbb{R}$ with the standard norm.

Now what's $L^\infty(X)$? It consists of functions bounded almost everywhere, but there are no nontrivial measure-zero sets. So L^∞ in fact consists of functions bounded everywhere. That is, $g \in L^\infty$ iff $|g(x)| < \infty$ and $|g(y)| < \infty$. So L^∞ is isomorphic to $\mathbb{R}^2$ as a vector space. The norm induced on $\mathbb{R}^2$ is clearly the supremum norm: $\|(a,b)\|_{L^\infty} = \max\{|a|, |b|\}$.

Now $\mathbb{R}^1$ is not isomorphic to $\mathbb{R}^2$, so $(L^1)^*$ is not isomorphic to L^∞. (Note the subtlety here: we've often said any L^∞ function gives a functional on L^1, but in this case the problem is that this correspondence is not $1-1$. Two bounded functions with the same value at x but different values on y will induce the exact same functional on L^1.)

In fact this works even if both singletons have infinite measure, but it's a little boring: in that case $L^1(X)$ is just the trivial vector space since any nontrivial function would have infinite integral. So the dual space is also the trivial vector space, while $L^\infty(X)$ is just isomorphic to $\mathbb{R}^2$ with the sup norm, and they are certainly not equal.

The third possibility is $\mu\{x\} = 0$ and $\mu\{y\} = \infty$. In this case things are a bit more subtle: notice that

$$\int_X |f| \, d\mu = \begin{cases} 0 & f(y) = 0, \\ \infty & f(y) \neq 0. \end{cases}$$

Thus it looks like $L^1(X)$ will be isomorphic to $\mathbb{R}$ since $f(x)$ is arbitrary. However remember that $f \in L^1$ only makes sense up to sets of measure zero, and $\{x\}$ is a set of measure zero. So up to equivalence the only function in L^1 is the zero function. That means $(L^1)^*$ is the zero functional as well. On the other hand, L^∞ is the set of bounded functions, up to almost-everywhere equivalence. So L^∞ is a one-dimensional vector space ($g(y)$ can be an arbitrary number, and $g(x)$ might as well be zero since the equivalence class is the same no matter what). In this case too, $(L^1)^* \neq L^\infty$.

4. *Suppose μ_1, μ_2, ν_1, and ν_2 are σ-finite measures, and that ν_1 is absolutely continuous with respect to μ_1 and ν_2 is absolutely continuous with respect to μ_2.*

   Prove that $\nu_1 \times \nu_2$ is absolutely continuous with respect to $\mu_1 \times \mu_2$, and prove that

   $$\left[\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}\right] = \left[\frac{d\nu_1}{d\mu_1}\right] \left[\frac{d\nu_2}{d\mu_2}\right].$$

   The problem implicitly assumes that μ_1 and ν_1 are defined on the same σ-algebra, and that μ_2 and ν_2 are defined on the same σ-algebra, since absolute continuity of one measure with respect to another only makes sense then. So the product measure σ-algebras both contain the smallest σ-algebra containing the measurable rectangles.

   Suppose E is a μ-measurable set with $(\mu_1 \times \mu_2)(E) = 0$. We want to prove that $(\nu_1 \times \nu_2)(E) = 0$. By Lemma 12.17, we know that for μ_1-almost all $x \in X$, $\mu_2(E_x) = 0$. For every such x, since ν_2 is absolutely continuous w.r.t. μ_2 we have $\nu_2(E_x) = 0$. Also since ν_1 is absolutely continuous w.r.t. μ_1, this happens ν_1-almost everywhere. Therefore by Lemma 12.18 we have

   $$(\nu_1 \times \nu_2)(E) = \int_X \xi_{E_x} \, d\nu_1$$

is the integral of a function which is zero almost everywhere, and hence is zero. This proves that $\nu_1 \times \nu_2$ is absolutely continuous with respect to $\mu_1 \times \mu_2$, so that the Radon-Nikodym derivative exists.

To prove the Radon-Nikodym derivative of a product is the product of the Radon-Nikodym derivatives, we again use Fubini's theorem.

To simplify the notation a bit, let $h = \frac{d[\nu_1 \times \nu_2]}{d[\mu_1 \times \mu_2]}$, let $f = \frac{d\nu_1}{d\mu_1}$, and let $g = \frac{d\nu_2}{d\mu_2}$. We want to prove that $h(x, y) = f(x)g(y)$.

Let E be any measurable set in $X \times Y$. Then by definition of the Radon-Nikodym derivative we have

$$(\nu_1 \times \nu_2)(E) = \int_{X \times Y} h\xi_E \, d(\mu_1 \times \mu_2).$$

But also by the Tonelli theorem (since both spaces are σ-finite) we have

$$\begin{aligned}(\nu_1 \times \nu_2)(E) &= \int_X \left[\int_Y \xi_E \, d\nu_2\right] d\nu_1 = \int_X [\nu_2(E_x)] \, d\nu_1 \\ &= \int_X \left[\int_{E_x} g \, d\mu_2\right] d\nu_1 = \int_X f \left[\int_Y g\xi_{E_x} \, d\mu_2\right] d\mu_1 \\ &= \int_X fg\xi_E \, d(\mu_1 \times \mu_2)\end{aligned}$$

using the Tonelli theorem again. Now since this is true for every measurable set E, we have

$$\int_{X \times Y} h\varphi \, d(\mu_1 \times \mu_2) = \int_{X \times Y} fg\varphi \, d(\mu_1 \times \mu_2)$$

for every nonnegative simple function φ. And thus by definition of the integral, we know it's true for every nonnegative function φ.

Thus we have two Radon-Nikodym derivatives, but we know that Radon-Nikodym derivatives are unique up to a set of $\mu_1 \times \mu_2$-measure zero. So $h = fg$, up to a set of $\mu_1 \times \mu_2$-measure zero.

5. *Let* X *be a metric space and* E *be a Borel subset of* X*. Let* $0 < \alpha < \beta < \infty$*.*

   Prove that if $m_\alpha(E) < \infty$ *then* $m_\beta(E) = 0$*.*

Prove that if $m_\beta(E) > 0$ *then* $m_\alpha(E) = \infty$.

Since $m_\alpha(E) < \infty$, we know $\lim_{\varepsilon\to 0} m_{\alpha,\varepsilon}(E) < \infty$, so for any $\varepsilon > 0$ sufficiently small, we have $m_{\alpha,\varepsilon}(E) < \infty$.

Choose an cover of E consisting of balls B_i with radius $r_i < \varepsilon$ so that

$$\sum_{i=1}^{\infty} r_i^\alpha < m_{\alpha,\varepsilon}(E) + \varepsilon;$$

this can always be done by definition of infimum.

Using this cover, we have

$$m_{\beta,\varepsilon}(E) \le \sum_{i=1}^{\infty} r_i^\beta = \sum_{i=1}^{\infty} r_i^{\beta-\alpha} r_i^\alpha \le \sum_{i=1}^{\infty} \varepsilon^{\beta-\alpha} r_i^\alpha \le \varepsilon^{\beta-\alpha}\left(m_{\alpha,\varepsilon}(E) + \varepsilon\right).$$

Now taking the limit of the right side as $\varepsilon \to 0$, and using the fact that $\lim_{\varepsilon\to 0} m_{\alpha,\varepsilon}(E) < \infty$, we have

$$\lim_{\varepsilon\to 0} m_{\beta,\varepsilon}(E) \le 0,$$

so that $m_\beta(E) = 0$ since it must be nonnegative.

The second statement is the contrapositive of the first since $0 \le m_\alpha(E) \le \infty$ for all E and α. So it's also true.