Math 6230 Homework #6 Solutions

1. Read all of Sections 10 (Derivatives) and 11 (Tangent bundle) of "Vector Calculus for Differential Geometry."

2. Consider the map $F\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^z$.

   (a) Compute $(F_*)_z$ at any point z in rectangular coordinates.

   Solution:

   Since $e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$, in rectangular coordinates we have $\mathbf{x} \circ F \circ \mathbf{x}^{-1}(x, y) = (e^x \cos y, e^x \sin y)$. Write (u, v) for the coordinates on the range; then we have $(u, v) = (e^x \cos y, e^x \sin y)$, so that

$$F_*\left(\frac{\partial}{\partial x}\Big|_{(x,y)}\right) = \frac{\partial u}{\partial x}(x,y)\,\frac{\partial}{\partial u}\Big|_{(u,v)} + \frac{\partial v}{\partial x}(x,y)\,\frac{\partial}{\partial v}\Big|_{(u,v)},$$
$$= e^x \cos y\,\frac{\partial}{\partial x}\Big|_{(e^x\cos y, e^x \sin y)} + e^x \sin y\,\frac{\partial}{\partial y}\Big|_{(e^x\cos y, e^x \sin y)},$$
$$F_*\left(\frac{\partial}{\partial y}\Big|_{(x,y)}\right) = -e^x \sin y\,\frac{\partial}{\partial x}\Big|_{(e^x\cos y, e^x \sin y)} + e^x \cos y\,\frac{\partial}{\partial y}\Big|_{(e^x\cos y, e^x \sin y)}.$$

   (b) Compute $(F_*)_z$ at any z in polar coordinates.

   Solution:

   We have $e^{re^{i\theta}} = e^{r\cos\theta + ir\sin\theta} = e^{r\cos\theta}e^{ir\sin\theta}$, so the map in polar coordinates is $\varphi \circ F \circ \varphi^{-1}(r, \theta) = (e^{r\cos\theta}, r\sin\theta)$.

   Hence F_* is

$$F_*\left(\frac{\partial}{\partial r}\Big|_{(r,\theta)}\right) = \cos\theta e^{r\cos\theta}\,\frac{\partial}{\partial r}\Big|_{(e^{r\cos\theta}, r\sin\theta)} + \sin\theta\,\frac{\partial}{\partial \theta}\Big|_{(e^{r\cos\theta}, r\sin\theta)},$$
$$F_*\left(\frac{\partial}{\partial \theta}\Big|_{(r,\theta)}\right) = -r\sin\theta e^{r\cos\theta}\,\frac{\partial}{\partial r}\Big|_{(e^{r\cos\theta}, r\sin\theta)} + r\cos\theta\,\frac{\partial}{\partial \theta}\Big|_{(e^{r\cos\theta}, r\sin\theta)}.$$

3. A vector field is a smooth map $V\colon M \to TM$ such that $V(p) \in T_pM$ for every p. Show that if M is an n-dimensional manifold with vector fields $V_1, \cdots V_n$ which are linearly independent at every point, then TM is isomorphic to the trivial bundle $M \times \mathbb{R}^n$; that is, there is a homeomorphism $\varphi\colon M \times \mathbb{R}^n \to TM$ such that for each fixed p, the restriction $\varphi\big|_{\{p\}\times\mathbb{R}^n} \to T_pM$ is a vector space isomorphism.

   Solution:

   Let $\varphi\colon M \times \mathbb{R}^n \to TM$ be the map

$$\varphi(p, v^1, \cdots, v^n) = \sum_{k=1}^{n} v^k V_k(p).$$

Since the vector fields are smooth, φ is smooth. For each fixed p it is a linear transformation of the vector spaces, since

$$\varphi(p, av^1, \cdots, av^n) + \varphi(p, bw^1, \cdots, bw^n) = \sum_{k=1}^{n} av^k V_k(p) + \sum_{k=1}^{n} bw^k V_k(p)$$
$$= \varphi(p, av^1 + bw^1, \cdots, av^n + bw^n).$$

Furthermore it is an isomorphism, since it is one-to-one: $\varphi(p, v^1, \cdots, v^n) = 0$ implies $\sum_{k=1}^n v^k V_k(p) = 0$ which implies that all v^k are zero, since the $V_k(p)$ are linearly independent.

The map is clearly surjective, since every vector in any T_pM can be written in terms of the vector fields V_k, and it is one-to-one since $\varphi(p, v^1, \cdots, v^n) = \varphi(q, w^1, \cdots, w^n)$ implies $p = q$ (since the only way a vector can be in T_pM and T_qM is if $p = q$) and also $v^k = w^k$ for all k.

Finally we can check that it is a homeomorphism by verifying that the inverse map (which exists by the previous paragraph) is continuous. One way to do this is to use a coordinate trivialization, since if a map is continuous in every coordinate chart, then it is continuous on the entire manifold. If ψ is a coordinate map on U and Ψ the trivialization of TU, with the vector fields in coordinates looking like $V_k(p) = \sum_{j=1}^n a_k^j(p) \frac{\partial}{\partial x^j}\Big|_p$ then

$$\Psi\left(\sum_{k=1}^{n} v^k V_k(p)\right) = \Psi\left(\sum_{k=1}^{n}\sum_{j=1}^{n} v^k a_k^j(p) \frac{\partial}{\partial x^j}\Big|_p\right)$$
$$= \left(x^1(p), \cdots, x^n(p), \sum_{k=1}^{n} v^k a_k^1(p), \cdots, \sum_{k=1}^{n} v^k a_k^n(p)\right),$$

so that

$$\varphi(p, v^1, \cdots, v^n) = \Psi^{-1}\left(x^1(p), \cdots, x^n(p), \sum_{k=1}^{n} v^k a_k^1(p), \cdots, \sum_{k=1}^{n} v^k a_k^n(p)\right),$$

and using this formula we can compute the inverse φ^{-1} in terms of Ψ and the inverse matrix of (a_j^i) (which always exists and is therefore continuous).

4. Show that if $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a function and $r \in \mathbb{R}$ is a regular value of F, then the map $G\colon \mathbb{R}^{2n+2} \to \mathbb{R}^2$ given by

$$G(x^1, \cdots, x^{n+1}, v^1, \cdots, v^{n+1}) = \left(F(x^1, \cdots, x^{n+1}), \sum_{k=1}^{n+1} \tfrac{\partial F}{\partial x^k}(x^1, \cdots, x^{n+1}) v^k\right)$$

has rank two everywhere. Conclude that $G^{-1}(r, 0)$ is a submanifold.

Show that the embedding $\iota\colon F^{-1}(r) \to \mathbb{R}^{n+1}$ induces an embedding $\iota_*\colon TF^{-1}(r) \to \mathbb{R}^{2n+2}$, and that the image $\iota_*[TF^{-1}(r)]$ is $G^{-1}(r, 0)$.

Solution:

The $2 \times 2(n+1)$ matrix DG is

$$DG(x^1, \cdots, x^{n+1}, v^1, \cdots, v^{n+1}) = \begin{pmatrix} \frac{\partial F}{\partial x^1} & \cdots & \frac{\partial F}{\partial x^{n+1}} & 0 & \cdots & 0 \\ \sum_k \frac{\partial^2 F}{\partial x^1 \partial x^k} v^k & \cdots & \sum_k \frac{\partial^2 F}{\partial x^n \partial x^k} v^k & \frac{\partial F}{\partial x^1} & \cdots & \frac{\partial F}{\partial x^{n+1}} \end{pmatrix}.$$

Since r is a regular value of F, we know that if $F(x^1, \cdots, x^{n+1}) = r$, then the vector $\left(\frac{\partial F}{\partial x^1} \quad \cdots \quad \frac{\partial F}{\partial x^{n+1}}\right)$ is nonzero, and so the two rows of DG are linearly independent. This is true everywhere on $G^{-1}(r, s)$, in particular when $s = 0$.

So $(r, 0)$ is a regular value of G, which implies that $G^{-1}(r, 0)$ is a smooth manifold of dimension $2(n+1) - 2 = 2n$.

Now the embedding ι automatically gives a smooth map $\iota_* \colon TF^{-1}(r) \to \mathbb{R}^{2n+2}$, so we just have to confirm that it is an embedding (i.e., that it is a immersion and that it is a homeomorphism onto its image).

First, suppose we have a coordinate chart $\varphi = (u^1, \cdots, u^n)$ on an open set U in $F^{-1}(r)$. Then the trivialization on TU is given by

$$\Phi^{-1}(u^1, \cdots, u^n, v^1, \cdots, v^n) = \sum_{k=1}^{n} v^k \frac{\partial}{\partial u^k}\Big|_{\varphi^{-1}(u^1, \cdots, u^n)}.$$

If ψ is the usual Cartesian chart on $\mathbb{R}^{n+1}$ and Ψ is the corresponding trivialization of $T\mathbb{R}^{n+1}$, then in these coordinates the map ι_* looks like

$$\Psi \circ \iota_* \circ \Phi^{-1}(u^1, \cdots, u^n, v^1, \cdots, v^n) = \Big(x^1(u^1, \cdots, u^n), \cdots, x^{n+1}(u^1, \cdots, u^n), \\ \sum_k v^k \frac{\partial x^1}{\partial u^k}, \cdots, \sum_k v^k \frac{\partial x^{n+1}}{\partial u^k}\Big)$$

(where I'm abbreviating $x^j \circ \iota \circ \phi^{-1}(u)$ by $x^j(u)$). Its derivative matrix in these coordinates is the $(2n+2) \times 2n$ matrix

$$D(\Psi \circ \iota_* \circ \Phi^{-1}) = \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^{n+1}}{\partial u^n} & \cdots & \frac{\partial x^{n+1}}{\partial u^n} & 0 & \cdots & 0 \\ \sum_k v^k \frac{\partial^2 x^1}{\partial u^1 \partial u^k} & \cdots & \sum_k v^k \frac{\partial^2 x^1}{\partial u^n \partial u^k} & \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^1}{\partial u^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_k v^k \frac{\partial^2 x^{n+1}}{\partial u^1 \partial u^k} & \cdots & \sum_k v^k \frac{\partial^2 x^{n+1}}{\partial u^n \partial u^k} & \frac{\partial x^{n+1}}{\partial u^1} & \cdots & \frac{\partial x^{n+1}}{\partial u^n} \end{pmatrix}.$$

Now this is a block matrix, and the upper left $(n+1) \times n$ matrix has rank n since ι itself is an immersion. Hence the lower right $(n+1) \times n$ matrix also has rank n, and thus the entire thing has rank $2n$.

Since ι_* is an immersion, it is an open map (that is, the image of an open set is open) by the inverse function theorem. Since ι_* is one-to-one, it is therefore an embedding.

Finally we verify that $\iota_*[TF^{-1}(r)] = G^{-1}(r,0)$. This comes from the fact that in any coordinate chart $\varphi = (u^1, \cdots, u^n)$, we have $\iota_*(\frac{\partial}{\partial u^j} = \sum_{k=1}^{n+1} \frac{\partial x^k}{\partial u^j} \frac{\partial}{\partial x^k}$, and the trivialization of $\mathbb{R}^{n+1}$ just comes from looking at the components of $\frac{\partial}{\partial x^k}$. Furthermore we have that

$$F\big(x^1(u^1, \cdots, u^n), \cdots, x^n(u^1, \cdots, u^n)\big) = r$$

for all u, and hence differentiating this equation with respect to u^j we get

$$\sum_{k=1}^{n+1} \frac{\partial F}{\partial x^k} \frac{\partial x^k}{\partial u^j} = 0.$$

Hence any vector in $\iota_*[T_pF^{-1}(r)]$ must be perpendicular to the gradient of F in $\mathbb{R}^{n+1}$. Conversely since $(\iota_*)_p$ is one-to-one as a linear transformation, its image must have the same dimension as its domain, which means it fills up all of $G^{-1}(r,0)$.

5. Consider the following vector fields in $\mathbb{R}^4$.

$$\begin{aligned} V_0 &= w\,\tfrac{\partial}{\partial w} + x\,\tfrac{\partial}{\partial x} + y\,\tfrac{\partial}{\partial y} + z\,\tfrac{\partial}{\partial z}, \\ V_1 &= -x\,\tfrac{\partial}{\partial w} + w\,\tfrac{\partial}{\partial x} - z\,\tfrac{\partial}{\partial y} + y\,\tfrac{\partial}{\partial z}, \\ V_2 &= -y\,\tfrac{\partial}{\partial w} + z\,\tfrac{\partial}{\partial x} + w\,\tfrac{\partial}{\partial y} - x\,\tfrac{\partial}{\partial z}, \\ V_3 &= -z\,\tfrac{\partial}{\partial w} - y\,\tfrac{\partial}{\partial x} + x\,\tfrac{\partial}{\partial y} + w\,\tfrac{\partial}{\partial z}. \end{aligned}$$

Show that they are orthonormal and that V_1, V_2, V_3 span T_pS^3 for every $p \in S^3$. Conclude that TS^3 is isomorphic to $S^3 \times \mathbb{R}^3$.

Solution: We obviously have

$$\langle V_0, V_0\rangle = \langle V_1, V_1\rangle = \langle V_2, V_2\rangle = \langle V_3, V_3\rangle = w^2 + x^2 + y^2 + z^2.$$

So at a point $(w, x, y, z) \in S^3$, they all have unit length. We can also easily compute

$$\begin{aligned} \langle V_0, V_1\rangle &= -wx + xw - yz + yz = 0, \\ \langle V_0, V_2\rangle &= -wy + xz + wy - xz = 0, \\ \langle V_0, V_3\rangle &= -wz - xy + xy + wz = 0, \\ \langle V_1, V_2\rangle &= xy + wz - wz - xy = 0, \\ \langle V_1, V_3\rangle &= xz - wy - xz + wy = 0, \\ \langle V_2, V_3\rangle &= yz - yz + wx - wx = 0. \end{aligned}$$

So the vectors are orthonormal at every point. Since $S^3 = F^{-1}(1)$ where $F(w, x, y, z) = w^2 + x^2 + y^2 + z^2$, we see from problem 3 that the tangent space T_pS^3 consists of those vectors orthogonal to the gradient vector V_0, and hence it is spanned everywhere by V_1, V_2, and V_3. Since these vectors are orthonormal, they are in particular linearly independent.

Hence problem 2 implies that TS^3 is a trivial bundle.