

Math 6230 Homework #5 Solutions

1. Read all of Section 9 (Vectors) of “Vector Calculus for Differential Geometry.”
2. In the parabolic coordinate system (σ, τ) defined by

$$(x, y) = \left(\sigma\tau, \frac{1}{2}(\tau^2 - \sigma^2)\right),$$

express the basis vectors $\frac{\partial}{\partial\sigma}$ and $\frac{\partial}{\partial\tau}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. (Refer to Section 5 for more on parabolic coordinates.)

Solution:

Recall that these equations were solved in Section 5 for σ and τ : in the coordinate chart defined by $\tau > 0, \sigma \in \mathbb{R}$, we have

$$\tau = \sqrt{y + \sqrt{x^2 + y^2}} \quad \text{and} \quad \sigma = \frac{x}{\sqrt{y + \sqrt{x^2 + y^2}}}.$$

We have

$$\begin{aligned} \frac{\partial}{\partial\sigma} &= \frac{\partial x}{\partial\sigma} \frac{\partial}{\partial x} + \frac{\partial y}{\partial\sigma} \frac{\partial}{\partial y} = \tau \frac{\partial}{\partial x} - \sigma \frac{\partial}{\partial y} \\ &= \sqrt{y + \sqrt{x^2 + y^2}} \frac{\partial}{\partial x} - \frac{x}{\sqrt{y + \sqrt{x^2 + y^2}}} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial\tau} &= \frac{\partial x}{\partial\tau} \frac{\partial}{\partial x} + \frac{\partial y}{\partial\tau} \frac{\partial}{\partial y} = \sigma \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial y} \\ &= \frac{x}{\sqrt{y + \sqrt{x^2 + y^2}}} \frac{\partial}{\partial x} + \sqrt{y + \sqrt{x^2 + y^2}} \frac{\partial}{\partial y}. \end{aligned}$$

3. Consider the function $f: \mathbb{C} \rightarrow \mathbb{R}$ defined by $f(z) = \text{Im}(z^3)$. For each $z \in \mathbb{C}$, let $\gamma_z(t) = e^{it}z^2$.

- (a) Compute $(f \circ \gamma_z)'(0)$ directly.

Solution:

Let's express $z = \rho e^{i\psi}$. Then $\gamma_z(t) = e^{it} \rho^2 e^{2i\psi}$. Then we have

$$(f \circ \gamma_z)(t) = \text{Im}(\rho^6 e^{3it+6i\psi}) = \rho^6 \sin(3t + 6\psi).$$

Thus

$$(f \circ \gamma_z)'(0) = 3\rho^6 \cos 6\psi.$$

- (b) Compute $f \circ \mathbf{x}^{-1}$, $\mathbf{x} \circ \gamma_z$, and $(f \circ \gamma_z)'(0)$ from them, using rectangular coordinates (x, y) .

Solution:

First,

$$f \circ \mathbf{x}^{-1}(x, y) = \text{Im}((x + iy)^3) = \text{Im}(x^3 + 3ix^2y - 3xy^2 - iy^3) = 3x^2y - y^3.$$

Second,

$$\mathbf{x} \circ \gamma_z(t) = (\rho^2 \cos(t + 2\psi), \rho^2 \sin(t + 2\psi)).$$

Thus

$$\begin{aligned} (f \circ \gamma_z)'(0) &= \frac{\partial(f \circ \mathbf{x}^{-1})}{\partial x} x'(0) + \frac{\partial(f \circ \mathbf{x}^{-1})}{\partial y} y'(0) \\ &= -6xy \rho^2 \sin 2\psi + (3x^2 - 3y^2) \rho^2 \cos 2\psi \\ &= -6\rho^6 \cos 2\psi \sin^2 2\psi + 3\rho^6(\cos^2 2\psi - \sin^2 2\psi) \cos 2\psi \\ &= 3\rho^6(\cos^3 2\psi - 3 \cos 2\psi \sin^2 2\psi) \\ &= 3\rho^6 \cos 6\psi. \end{aligned}$$

(c) Compute $f \circ \mathbf{u}^{-1}$, $\mathbf{u} \circ \gamma_z$, and $(f \circ \gamma_z)'(0)$ from them, using polar coordinates (r, θ) .

Solution:

We have $f \circ \mathbf{u}^{-1}(r, \theta) = \text{Im}(r^3 e^{3i\theta}) = r^3 \sin 3\theta$, and

$$\mathbf{u} \circ \gamma_z(t) = (\rho^2, t + 2\psi).$$

Thus

$$\begin{aligned} (f \circ \gamma_z)'(0) &= \frac{\partial(f \circ \mathbf{u}^{-1})}{\partial r} r'(0) + \frac{\partial(f \circ \mathbf{u}^{-1})}{\partial \theta} \theta'(0) \\ &= 3r^2 \sin 3\theta \cdot 0 + 3r^3 \cos 3\theta \cdot 1 \\ &= 3\rho^6 \cos 6\psi. \end{aligned}$$

4. Express the vector X , given in rectangular coordinates by

$$X_{(x,y)} = (3x^2 - y) \frac{\partial}{\partial x} \Big|_{(x,y)} + y^3 \frac{\partial}{\partial y} \Big|_{(x,y)},$$

in polar coordinates.

Solution:

Just use the transformation formulas:

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}. \end{aligned}$$

Then we have

$$\begin{aligned} X &= \left(3r^2 \cos^2 \theta - r \sin \theta \right) \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \\ &\quad + \left(r^3 \sin^3 \theta \right) \left(\sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right) \\ &= \left(3r^2 \cos^3 \theta - r \sin \theta \cos \theta + r^3 \sin^4 \theta \right) \frac{\partial}{\partial r} \\ &\quad + \left(-3r \cos^2 \theta \sin \theta + \sin^2 \theta + r^2 \sin^3 \theta \cos \theta \right) \frac{\partial}{\partial \theta}. \end{aligned}$$

5. Express the vector X , given in elliptic coordinates (μ, ν) by

$$X_{(\mu, \nu)} = \sinh \mu \cos \nu \left. \frac{\partial}{\partial \mu} \right|_{(\mu, \nu)} + \cosh \mu \sin \nu \left. \frac{\partial}{\partial \nu} \right|_{(\mu, \nu)},$$

in rectangular coordinates.

Solution:

Using the formulas $x = \cosh \mu \cos \nu$ and $y = \sinh \mu \sin \nu$, the transformation is

$$\begin{aligned} X &= \sinh \mu \cos \nu \frac{\partial}{\partial \mu} + \cosh \mu \sin \nu \frac{\partial}{\partial \nu} \\ &= \sinh \mu \cos \nu \left(\frac{\partial x}{\partial \mu} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \mu} \frac{\partial}{\partial y} \right) + \cosh \mu \sin \nu \left(\frac{\partial x}{\partial \nu} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \nu} \frac{\partial}{\partial y} \right) \\ &= \sinh \mu \cos \nu \left(\sinh \mu \cos \nu \frac{\partial}{\partial x} + \cosh \mu \sin \nu \frac{\partial}{\partial y} \right) \\ &\quad + \cosh \mu \sin \nu \left(-\cosh \mu \sin \nu \frac{\partial}{\partial x} + \sinh \mu \cos \nu \frac{\partial}{\partial y} \right) \\ &= (\sinh^2 \mu \cos^2 \nu - \cosh^2 \mu \sin^2 \nu) \frac{\partial}{\partial x} + 2 \sinh \mu \cosh \mu \cos \nu \sin \nu \frac{\partial}{\partial y} \\ &= (\sinh^2 \mu (1 - \sin^2 \nu) - \cosh^2 \mu (1 - \cos^2 \nu)) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} \\ &= (x^2 - y^2 - 1) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y}. \end{aligned}$$

6. The usual spherical coordinates are related to rectangular coordinates by $(x, y, z) = (\rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi, \rho \cos \theta)$. Express the basis vectors $\frac{\partial}{\partial \rho}$, $\frac{\partial}{\partial \theta}$, and $\frac{\partial}{\partial \phi}$ in terms of the basis vectors $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$. Then solve to get the rectangular basis vectors in terms of the spherical basis vectors.

Solution:

We have

$$\begin{aligned} \frac{\partial}{\partial \rho} &= \frac{\partial x}{\partial \rho} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \rho} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \rho} \frac{\partial}{\partial z} \\ &= \sin \theta \cos \phi \frac{\partial}{\partial x} + \sin \theta \sin \phi \frac{\partial}{\partial y} + \cos \theta \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial}{\partial z} \\ &= \rho \cos \theta \cos \phi \frac{\partial}{\partial x} + \rho \cos \theta \sin \phi \frac{\partial}{\partial y} - \rho \sin \theta \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi} \frac{\partial}{\partial z} \\ &= -\rho \sin \theta \sin \phi \frac{\partial}{\partial x} + \rho \sin \theta \cos \phi \frac{\partial}{\partial y}. \end{aligned}$$

We have $\rho \sin \theta \frac{\partial}{\partial \rho} + \cos \theta \frac{\partial}{\partial \theta} = \rho \left(\cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y} \right)$ and $\frac{\partial}{\partial \phi} = \rho \sin \theta \left(-\sin \phi \frac{\partial}{\partial x} + \cos \phi \frac{\partial}{\partial y} \right)$,
so that

$$\frac{\partial}{\partial x} = \frac{\cos \theta}{\rho} \left(\rho \sin \theta \frac{\partial}{\partial \rho} + \cos \theta \frac{\partial}{\partial \theta} \right) - \frac{\sin \phi}{\rho \sin \theta} \frac{\partial}{\partial \phi}$$

and

$$\frac{\partial}{\partial y} = \frac{\sin \theta}{\rho} \left(\rho \sin \theta \frac{\partial}{\partial \rho} + \cos \theta \frac{\partial}{\partial \theta} \right) + \frac{\cos \phi}{\rho \sin \theta} \frac{\partial}{\partial \phi}.$$

Finally we get

$$\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial \rho} - \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta}.$$