Math 6230 Homework #4 Solutions

1. Read all of Section 8 of "Vector Calculus for Differential Geometry."

2. Consider the following map from $\mathbb{R}^3$ to 2×2 matrices (identified with $\mathbb{R}^4$:

$$F(\alpha,\beta,\psi) = \begin{bmatrix} \cosh\psi\cos\alpha + \sinh\psi\cos\beta & \cosh\psi\sin\alpha + \sinh\psi\sin\beta \\ -\cosh\psi\sin\alpha + \sinh\psi\sin\beta & \cosh\psi\cos\alpha - \sinh\psi\cos\beta \end{bmatrix}.$$

Show that the image of F is a subset of $SL(2)$. Compute the rank of F, and find a maximal open subset of $\mathbb{R}^3$ on which F could be used as a coordinate chart. What portion of $SL(2)$ is covered by this chart?

Solution:

To show the image of F is a subset of $SL(2)$, we just have to compute the determinant of $F(\alpha,\beta,\psi)$, which is obviously

$$\begin{aligned}(\cosh\psi\cos\alpha + \sinh\psi\cos\beta)&(\cosh\psi\cos\alpha - \sinh\psi\cos\beta) \\ &- (\cosh\psi\sin\alpha + \sinh\psi\sin\beta)(-\cosh\psi\sin\alpha + \sinh\psi\sin\beta) \\ &= \cosh^2\psi\cos^2\alpha - \sinh^2\psi\cos^2\beta + \cosh^2\psi\sin^2\alpha - \sinh^2\psi\sin^2\beta \\ &= \cosh^2\psi - \sinh^2\psi = 1,\end{aligned}$$

no matter what α, β, and ψ are.

The rank of F is computed in terms of DF: we think of the image of F as being $\mathbb{R}^4$, so that

$$DF(\alpha,\beta,\psi) = \begin{bmatrix} -\cosh\psi\sin\alpha & -\sinh\psi\sin\beta & \sinh\psi\cos\alpha + \cosh\psi\cos\beta \\ \cosh\psi\cos\alpha & \sinh\psi\cos\beta & \sinh\psi\sin\alpha + \cosh\psi\sin\beta \\ -\cosh\psi\cos\alpha & \sinh\psi\cos\beta & -\sinh\psi\sin\alpha + \cosh\psi\sin\beta \\ -\cosh\psi\sin\alpha & \sinh\psi\sin\beta & \sinh\psi\cos\alpha - \cosh\psi\cos\beta \end{bmatrix}.$$

Computing the rank directly is a bit of a mess, though it could certainly be done. It's a little easier if we remember that the elementary row reduction operations do not change the rank. So we can replace the fourth row R_4 with $\frac{1}{2}(R_4 - R_1)$, and replace the third row R_3 with $\frac{1}{2}(R_3 + R_2)$. We get

$$DF(\alpha,\beta,\psi) \begin{bmatrix} -\cosh\psi\sin\alpha & -\sinh\psi\sin\beta & \sinh\psi\cos\alpha + \cosh\psi\cos\beta \\ \cosh\psi\cos\alpha & \sinh\psi\cos\beta & \sinh\psi\sin\alpha + \cosh\psi\sin\beta \\ 0 & \sinh\psi\cos\beta & \cosh\psi\sin\beta \\ 0 & \sinh\psi\sin\beta & -\cosh\psi\cos\beta \end{bmatrix}.$$

We can further reduce the first two rows in the obvious way:

$$DF(\alpha,\beta,\psi) \begin{bmatrix} -\cosh\psi\sin\alpha & 0 & \sinh\psi\cos\alpha \\ \cosh\psi\cos\alpha & 0 & \sinh\psi\sin\alpha \\ 0 & \sinh\psi\cos\beta & \cosh\psi\sin\beta \\ 0 & \sinh\psi\sin\beta & -\cosh\psi\cos\beta \end{bmatrix}.$$

This is now much more manageable. Compute the determinant of the first three rows and get

$$\sinh^2\psi\cosh\psi\cos\beta\cos^2\alpha + \cosh\psi\sinh^2\psi\cos\beta\sin^2\alpha = \sinh^2\psi\cosh\psi\cos\beta.$$

Hence the only possible way the rank could be less than three is if either $\sinh\psi = 0$ (which forces $\psi = 0$) or if $\cos\beta = 0$ (which forces $\sin\beta = \pm 1$).

First suppose $\psi = 0$. Then

$$DF(\alpha,\beta,0)\begin{bmatrix} -\sin\alpha & 0 & 0 \\ \cos\alpha & 0 & 0 \\ 0 & 0 & \sin\beta \\ 0 & 0 & -\cos\beta \end{bmatrix}$$

which clearly has rank two, since the first and third columns are linearly independent while the second column is zero.

Next suppose $\cos\beta = 0$, so that $\sin\beta = \pm 1$. Suppose also that $\psi \neq 0$. Then

$$DF(\alpha,\beta,\psi)\begin{bmatrix} -\cosh\psi\sin\alpha & 0 & \sinh\psi\cos\alpha \\ \cosh\psi\cos\alpha & 0 & \sinh\psi\sin\alpha \\ 0 & 0 & \pm\cosh\psi \\ 0 & \pm\sinh\psi & 0 \end{bmatrix}.$$

Compute the determinant of the matrix formed by the first, second, and fourth rows, and we get

$$\pm\sinh^2\psi\cosh\psi \neq 0.$$

So the only time DF has nonmaximal rank is when $\psi = 0$.

Now if we want to find an acceptable open set U, on which $(F^{-1}, F[U])$ could form a coordinate chart, we need F to have maximal rank on U (so that its inverse is smooth, by the implicit function theorem) and we need F to be one-to-one. Hence we can have either a subset of $\{(\alpha,\beta,\psi)\,|\,\psi > 0\}$ or a subset of $\{(\alpha,\beta,\psi)\,|\,\psi < 0\}$ but not both.

Let's choose $\psi > 0$. We've computed the rank already, so we just need to make sure F is one-to-one. Now clearly it's not: α and β both only enter through sines and cosines, so periodicity requires us to restrict the domain of α and β to some open 2π-sized interval. So let's consider

$$U = (0, 2\pi) \times (0, 2\pi) \times (0, \infty).$$

To see if this is acceptable, we just have to solve

$$F(\alpha,\beta,\psi) = F(\gamma,\delta,\eta)$$

where $\alpha,\gamma,\beta,\delta \in (0, 2\pi)$ and ψ and η are both positive.

Adding and subtracting the upper left and lower right entries, then adding and subtracting the upper right and lower left entries, we get the equations

$$\begin{aligned}\cosh\psi\cos\alpha &= \cosh\eta\cos\gamma,\\ \sinh\psi\cos\beta &= \sinh\eta\cos\delta,\\ \cosh\psi\sin\alpha &= \cosh\eta\sin\gamma,\\ \sinh\psi\sin\beta &= \sinh\eta\sin\delta.\end{aligned}$$

Square the first and third equations and add them: we get $\cosh^2\psi = \cosh^2\eta$ so that $\psi = \eta$ or $\psi = -\eta$. But since both must be positive, we have $\psi = \eta$. Hence we know $\cos\alpha = \cos\gamma$ and $\sin\alpha = \sin\gamma$, and since α and γ are both in the same 2π interval, we conclude $\alpha = \gamma$. Similarly $\beta = \delta$.

So F is one-to-one and smooth with smooth inverse, so the image of U is a coordinate chart.

Now we ask what the image of U is, and the easiest thing to do is to look at what the complement is. First we note that every matrix in $SL(2)$ comes from $F(\alpha,\beta,\psi)$ for some coordinates. This comes from solving the equations explicitly: we get $2\sinh\psi = \sqrt{(w-z)^2+(x+y)^2}$ and $wz - xy = 1$ implies $2\cosh\psi = \sqrt{(w-z)^2+(x+y)^2+4} = \sqrt{(w+z)^2+(x-y)^2}$. Since $(w+z)^2+(x-y)^2 = 2\cosh\psi$ we get that $w+z = 2\cosh\psi\cos\alpha$ and $x-y = 2\cosh\psi\sin\alpha$ for some uniquely determine $\alpha \in [0,2\pi)$, and similarly $\beta \in [0,2\pi)$ is uniquely determined unless $\psi = 0$ (in which case any β works).

So the matrices which are not in the coordinate chart are those coming from $\alpha = 0$, which just means $w+z > 0$ and $x = y$; those coming from $\beta = 0$, which means $x = -y$ and $w - z > 0$; and those coming from $\psi = 0$, which gives the orthogonal 2×2 matrices of the form $\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$.

3. What is a more familiar description of $O(1)$?

Solution:

$O(1)$ consists of the 1×1 matrices with $A^TA = 1$, which is just the set of numbers x such that $x^2 = 1$. So it's the two-point space $S^0 = \{-1, 1\}$.

4. What is the dimension of the Grassmann manifold $Gr(k,n)$ in terms of k and n? Explain briefly.

Solution:

In general we find a coordinate chart for $Gr(k,n)$ in a neighborhood of some plane by rotating $\mathbb{R}^n$ so that the k-plane is spanned by the unit vectors $e_1, e_2, \cdots, e_k$. Then some open neighborhood of this plane is covered by the planes spanned by vectors

$$\begin{aligned} v_1 &= e_1 + a_{1,k+1}e_{k+1} + \cdots + a_{1,n}e_n\\ v_2 &= e_2 + a_{2,k+1}e_{k+1} + \cdots + a_{2,n}e_n\\ &\vdots\\ v_k &= e_k + a_{k,k+1}e_{k+1} + \cdots + a_{k,n}e_n,\end{aligned}$$

and all these planes are distinct for different parameters a_{ij}, while any k-plane with a nontrivial projection onto the first k components can be expressed uniquely in this form.

The number of parameters required is $k(n-k)$, so this is the dimension of the Grassmann manifold $Gr(k,n)$.

5. Consider the Stiefel manifold $V(2,3)$ consisting of pairs of orthonormal vectors in $\mathbb{R}^3$. Find a coordinate chart for $V(2,3)$ in a neighborhood of $\{(1,0,0),(0,1,0)\}$, following the advice in the notes. (First pick a point of the unit sphere, then pick a point in the unit circle in the plane orthogonal to the point you picked before.) What points of the Stiefel manifold are not covered by this chart? There are multiple answers; you can use either spherical coordinates or stereographic coordinates, for example.

   Solution:

   Take spherical coordinates $(x,y,z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, with $\theta \in (0,\pi)$ and $\phi \in (-\pi,\pi)$. Then the given unit vector $(1,0,0)$ corresponds to $\theta = \frac{\pi}{2}$ and $\phi = 0$, and the excluded vectors are anything coming from $\theta = 0$ or $\theta = \pi$, or from $\phi = \pi$. These correspond to the vectors $(0,0,1)$, $(0,0,-1)$, and $(-\sin\theta, 0, \cos\theta)$ for $0 < \theta < \pi$.

   For each of the included vectors $u = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, two unit vectors orthogonal to each other and the given one are given by

   $$v = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \qquad \text{and} \qquad w = (-\sin\phi, \cos\phi, 0);$$

   these are the directions of increasing θ and ϕ respectively.

   So the unit circle in the plane orthogonal to u is traversed by $\cos\alpha v + \sin\alpha w$ for $\alpha \in [0, 2\pi)$. When $\theta = \frac{\pi}{2}$ and $\phi = 0$, we get $u = (1,0,0)$, $v = (0,0,-1)$, and $w = (0,1,0)$. So we want the second vector to be w, which corresponds to $\alpha = \frac{\pi}{2}$. Hence our coordinate chart for the circle should include $\frac{\pi}{2}$; we might as well pick $\alpha \in (-\frac{\pi}{2}, \frac{3\pi}{2})$.

   Hence our coordinates are $(\theta, \phi, \alpha) \in (0,\pi) \times (0, 2\pi) \times (-\frac{\pi}{2}, \frac{3\pi}{2})$, and the orthonormal vectors not included are any pairs with first vector $(-\sin\theta, 0, \cos\theta)$ for $\theta \in [0,\pi]$, or any pairs of the form $\{u, -w\}$, i.e., any pairs that look like

   $$\{(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), (\sin\phi, -\cos\phi, 0)\}.$$