Math 6230 Homework #3 Solutions
due Wednesday, September 23

1. Read all of Section 7 (First examples of manifolds).

2. Prove using polygonal representations the following:

   (a) that $\mathbb{T}^2 \# \mathbb{P}^2 = \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$;

   Solution:

   These things take forever to draw, and since you've all done them, I'll just list the words and the cuts.

   Start with a hexagon representing $\mathbb{T}^2 \# \mathbb{P}^2$, with word given by $aba^{-1}b^{-1}cc$.

   Cut off a triangle from the end of the first copy of a to the beginning of the second copy of c, and call this direction d. In this way get a pentagon $ba^{-1}b^{-1}cd^{-1}$ and a triangle adc. Flip the triangle over and glue it along c to get a hexagon $bab^{-1}dad$.

   Now from the point that ends a and starts d, draw a segment to the point that ends a and ends b. Call this side e. Cut along e to get two quadrilaterals $aeb^{-1}d$ and $ae^{-1}db$. Flip the second one and glue back together along a to get a polygon $eeb^{-1}db^{-1}d^{-1}$, which we recognize as a projective plane glued to a Klein bottle.

   Now the second part says that a Klein bottle is the connected sum of two projective planes, so we are done once we finish part (b).

   (b) that $\mathbb{P}^2 \# \mathbb{P}^2$ is the Klein bottle (described by $aba^{-1}b$).

   Solution:

   Write the Klein bottle as a square $aba^{-1}b$. Cut along the diagonal (let's say from the end of a and end of b, to the start of a and end of b) and call this direction c. We get two triangles acb^{-1} and abc. Flip the second one and glue together along b, and we end up with the square $aacc$, which is two projective planes.

3. Consider the compact surfaces described by the following words. Using the reductions described in class, classify them as either a sphere, the connected sum of n tori, or the connected sum of n projective planes.

   (a) $abcabc$

   Solution:

   The easy way to do it is to just notice that abc is a single side d, and the polygon is just dd. Hence it's $\mathbb{P}^2$.

   (b) $abacb^{-1}dc^{-1}d$

   We first try and peel off projective planes. Notice we have two sides a in the same direction, so we cut from the end of one copy of side a to the other (which is a triangle), flip it, and glue it back along a. We get a polygon $eebd^{-1}cd^{-1}bc^{-1}$. So we can remove $\mathbb{P}^2$ and just work with the hexagon $bd^{-1}cd^{-1}bc^{-1}$.

   Again we see two sides in the same direction, so we cut from one end of b to the other end of b. Flip the triangle we get and glue it back along b. We get $ffd^{-1}cd^{-1}c$. So we can pull off another projective plane, and what's left is $d^{-1}cd^{-1}c$. Again we can just define $g = d^{-1}c$ and get that what's left over is gg. So the full simplification is $eeffgg$, which is $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$.

(c) $abcde^{-1}c^{-1}da^{-1}b^{-1}e^{-1}$

Solution:

Again we first look for projective planes. Cut from one end of d to the other, and get a square, then flip the square and glue it back along d. We get the word $abcceffa^{-1}b^{-1}e^{-1}$. We get the expected projected plane ff and a free one from cc. So we pull them both out, and what's left is $abea^{-1}b^{-1}e^{-1}$.

Now all the sides seem to be going in opposite directions, so we use the torus-finder. Cut from one end of a to another, to get a triangle bef and a pentagon $fa^{-1}b^{-1}e^{-1}a$. Glue along e, and we end up with $afa^{-1}b^{-1}f^{-1}b$.

Now cut from one end of f to another and glue the triangle along b. We get $a^{-1}afg^{-1}f^{-1}g$.

The a terms cancel out and we're left with $fg^{-1}f^{-1}g$, which we recognize as a torus.

So the space is $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{T}^2$. Using part (2a), we can rewrite this as

$$\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2.$$

4. A two-dimensional manifold-with-boundary is defined to be a second-countable Hausdorff topological space M such that for each point $p \in M$, there is an open set $U \subset M$ containing p and a homeomorphism $\phi\colon U \to \mathbb{R}^2$ such that $\phi(p) = (0,0)$ and the image of ϕ is either all of $\mathbb{R}^2$ or the closed upper half-plane $H = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$. The point p is called an *interior point* if $\phi[U] = \mathbb{R}^2$, and it's called a *boundary point* if $\phi[U] = H$. The *boundary of* M is denoted by ∂M and consists of all boundary points.

Show that the boundary of a two-dimensional manifold-with-boundary is a one-dimensional manifold (without boundary). This fact is frequently summarized as "the boundary of a boundary is empty."

Solution A subspace of a Hausdorff space is Hausdorff, and a subspace of a second-countable space is second-countable. So the only thing we need to show is that every point in the boundary has a neighborhood homeomorphic to $\mathbb{R}$. The obvious way to do this is to use the homeomorphism we already have onto the half-plane.

Let p be any point on the boundary. Then there is an open set $U \subset M$ with $p \in U$ and a homeomorphism $\varphi\colon U \to H$. By definition of the subspace topology, $V = \partial M \cap U$ is open in ∂M. We want to show that the image of $\varphi|_V$ is the set $\mathbb{R} \times \{0\} = \{(x,0) | x \in \mathbb{R}$, which is obviously homeomorphic to $\mathbb{R}$, and then we'll be done.

First show that every point $(x,0)$ is $\varphi(q)$ for some $q \in \partial M \cap U$. Obviously $(x,0) = \varphi(q)$ for some $q \in U$ since φ is surjective onto the half-plane, so the only question is whether $q \in \partial M$. To show this, just define $\tilde{\varphi}\colon U \to H$ by

$$\tilde{\varphi}(r) = \varphi(r) - (x,0).$$

Since φ is a homeomorphism, so is $\tilde{\varphi}$, and clearly its image is H. Also $\tilde{\varphi}(q) = (x,0) - (x,0) = (0,0)$, which is the only other thing we need. Thus $q = \varphi^{-1}(x,0)$ is a boundary point. Since x was arbitrary, $\varphi[U \cap \partial M]$ contains $\mathbb{R} \times \{0\}$.

Next show that every (x,y) for $y > 0$ is *not* a boundary point. Let $\hat{H}$ denote the open upper half-plane $\{(x,y) | y > 0\}$; then $\hat{H}$ is homeomorphic to $\mathbb{R}^2$, using for example the correspondence $\eta\colon H \to \mathbb{R}^2$ given by $\eta(x,y) = (x, y - 1/y)$. Now $W = \varphi^{-1}(\hat{H})$ is an open set in M, and $\psi\colon W \to \mathbb{R}^2$ given by $\psi(r) = \eta(\varphi(r)) - \eta(x,y)$ is a homeomorphism which takes $\varphi^{-1}(x,y)$ to the origin. So $\varphi^{-1}(x,y)$ is an interior point if $y > 0$, and thus $\varphi[U \cap \partial M]$ contains only $\mathbb{R} \times \{0\}$.

Hence $\varphi|_{\partial M \cap U}$ is the desired homeomorphism onto $\mathbb{R}$, and we are done.

5. If M is a two-dimensional compact topological manifold with boundary, show that the boundary consists of finitely many disjoint circles.

   Hint: one version of compactness states that every sequence has a convergent subsequence. If there were infinitely many circles, you could take a point on each one and find a convergent subsequence; then the limiting point has infinitely many boundary points in any neighborhood of it. Why is this impossible? A similar argument shows that no component of the boundary can be homeomorphic to $\mathbb{R}$.

   Solution: There are a few ways to do this. One is to prove that ∂M is a closed subset of M, and therefore it is compact. Another is to take any open cover of ∂M and show directly that there is a finite subcover. Or one can just follow the hint.

   Following the hint, suppose that ∂M is *not* does not consist of finitely many circles. Then either one component of the boundary is homeomorphic to $\mathbb{R}$, or all components are circles but there are infinitely many.

   In case one component Σ is homeomorphic to $\mathbb{R}$ by some map $\gamma\colon \Sigma \to \mathbb{R}$, we can pick a sequence in $\mathbb{R}$ that doesn't converge (for example, the positive integers) and look at its inverse image in M, say $p_n = \varphi^{-1}(n)$. Since M is compact, there is a convergent subsequence $p_{n_k} \to p \in M$. Now this limit is either an interior point or a boundary point.

   If it's a boundary point, find an open set $U \ni p$ and a homeomorphism $\varphi\colon U \to H$ with $\varphi(p) = (0,0)$. Since $p_{n_k} \to p$, there is some K such that $k \geq K$ implies $p_{n_k} \in U$. Since φ is a homeomorphism, $\varphi(p_{n_k}) \to (0,0)$ in H. Now $(0,0)$ is a boundary point, and since each $\varphi(p_{n_k})$ is a boundary point as well, p must be on the same boundary component as p_{n_k}. So actually p_{n_k} converges to p in Σ, a contradiction.

   If p is an interior point, then we choose a coordinate chart (φ, U) around p mapping U to $\mathbb{R}^2$ and p to the origin. Again $\varphi(p_{n_k})$ is in $\varphi(U)$ for k large, but no boundary point can be contained in U since every point of U has a neighborhood homeomorphic to $\mathbb{R}^2$. Contradiction again.

   So Σ is not homeomorphic to $\mathbb{R}$.

   Similarly if we assume there are infinitely many circles in the boundary, we can take a point q_n from each one and find a subsequence q_{n_k} that converges to some point q of M. The limit q cannot be an interior point (since an $\mathbb{R}^2$ coordinate neighborhood of it would have to contain boundary points). If it's a boundary point, then a coordinate neighborhood has to map the points q_{n_k} to boundary points for k large enough, and that implies every q_{n_k} for k large enough is on the same boundary component. Contradiction.

Hence there must be finitely many circles.

6. Show that every compact two-dimensional manifold-with-boundary is either a sphere, the connected sum of n tori, or the connected sum of n projective planes, with a finite number of open discs removed. (Hint: If you glue discs onto a manifold-with-boundary along the boundary components, what do you get?)

   Solution: If we glue a closed disc onto a boundary circle of M, we get another manifold with boundary which has one fewer boundary component. So after finitely many such gluings, we get a manifold with boundary whose boundary components are empty; i.e., a genuine manifold.

   The resulting manifold is still compact, since it is a union of finitely many compact spaces (the original manifold with boundary is a subset of it, and each closed disc is a subset).

   We already know that the resulting compact manifold must be either S^2 or the connected sum of finitely many tori or projective planes, so we are done.

   (If we wanted to, we could prove this representation is unique, by proving that every manifold with boundary has a triangulation and then generalizing the Euler characteristic.)