Math 6230 Homework #2 Solutions

1. Read all of Sections 5 and 6 of "Vector Calculus for Differential Geometry."

2. "Bipolar" coordinates are defined in terms of distance to two given points. Suppose the two points are $(1,0)$ and $(-1,0)$, and we define a coordinate system by the formulas

$$(r,s) = \left(\sqrt{(x-1)^2+y^2}, \sqrt{(x+1)^2+y^2}\right). \tag{1}$$

   (a) Solve for $(x,y)$ in terms of $(r,s)$.

   **Solution:** We have $s^2 - r^2 = (x+1)^2 - (x-1)^2 = 4x$, so that $x = \frac{s^2-r^2}{4}$. From there, we get

$$y = \pm\frac{1}{4}\sqrt{-16+8(r^2+s^2)-(r^2-s^2)^2}.$$

   (b) Sketch the curves of fixed $r$ in the $xy$-plane (as done in the notes for polar, parabolic, and elliptic coordinates). Also sketch those of fixed $s$. You can use Maple or other software for this.

   **Solution:** We clearly have $r \geq 0$ and $s \geq 0$ by the definition of $r$ and $s$ as distances. Now it's obvious from the formulas that curves of constant $r$ are portions of circles centered at $(1,0)$, while curves of constant $s$ are portions of circles centered at $(-1,0)$. Sketching parametrically a few curves, we get the picture in Figure 1.

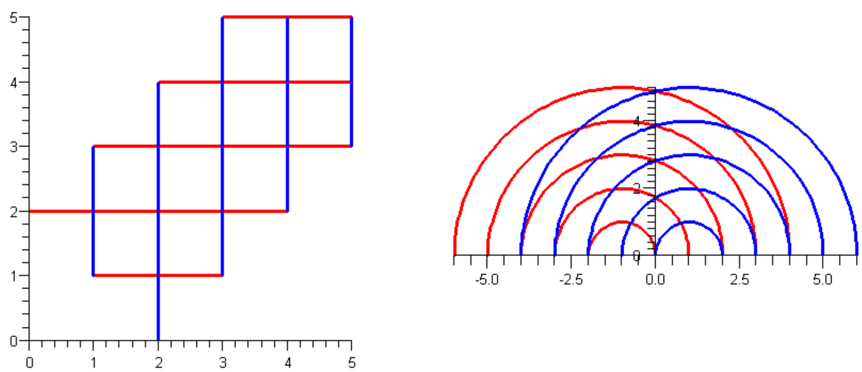



Figure 1: Coordinate curves in the $rs$-plane on the left, and their image in the $xy$-plane on the right.

   (c) Describe a maximal open set on which the map $(x,y) \mapsto (r,s)$ is smooth and invertible (in the $xy$-plane). Then describe the image of this set in the $rs$-plane.

   **Solution:** The Jacobian derivative of the transformation $(x,y) \mapsto (r,s)$ is

$$J = \frac{\partial r}{\partial x}\frac{\partial s}{\partial y} - \frac{\partial r}{\partial y}\frac{\partial s}{\partial y} = -\frac{2y}{\sqrt{(x+1)^2+y^2}\,\sqrt{(x-1)^2+y^2}}.$$

This is nonzero and nonsingular as long as $y > 0$ or $y < 0$, which divides the plane into two disjoint open sets.

In addition, since we can solve algebraically for $(x, y)$ in terms of $(r, s)$ up to the ambiguity in the sign of $y$, we know that we can use the entire upper half plane (or the entire lower half-plane) for $(x, y)$ and obtain an invertible transformation, which is therefore smooth.

There are a couple of ways to find the image of the upper half plane in the $rs$-plane. One is to use the triangle inequality: if $p = (1, 0)$ and $q = (-1, 0)$, then for any point $z = (x, y)$, we have $r = d(z, p)$ and $s = d(z, q)$. Therefore for any point $z$, we must have

$$d(z, p) \leq d(z, q) + d(q, p) = d(z, q) + 2,$$

which translates into $r \leq s + 2$. Similarly we must have $s \leq r + 2$. Finally we must have $2 \leq r + s$. These inequalities define a region in the positive quadrant of the $rs$-plane, bounded by three lines; the region we want is the interior of this, since $y > 0$ means that the three points $p$, $q$, and $z$ can never lie on a line.

The other way to do this is to simply use the equation for $y$ in terms of $r$ and $s$, and find conditions under which the discriminant is positive:

$$-16 + 8(r^2 + s^2) - (r^2 - s^2)^2 > 0,$$
$$(2 + r + s)(2 - r - s)(2 - r + s)(2 + r - s) < 0,$$

and we obtain from this (and the requirement $r > 0$, $s > 0$) that the region must be as shown in Figure 2.

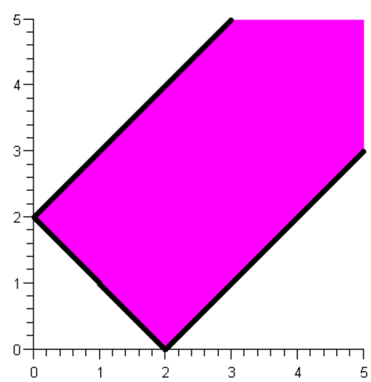



Figure 2: The image of the upper half of the $xy$-plane, in the $rs$-plane.

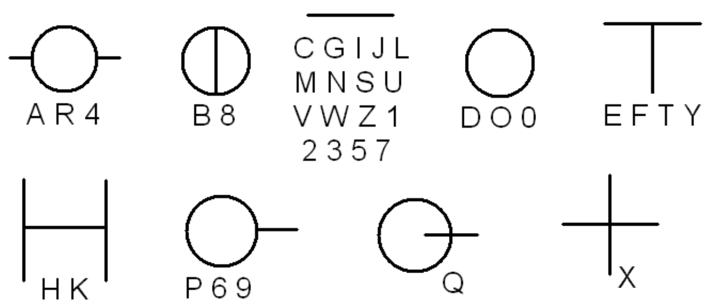
3. Consider the alphanumeric keyboard shown below.

   Classify these into disjoint sets based on homeomorphism type. You don't need to explain your reasoning.

   **Solution:**

A B C D E F G
H I J K L M
N O P Q R S T
U V W X Y Z
1 2 3 4 5
6 7 8 9 0

We reduce each letter to some kind of "prototype" shape. The only ambiguity is whether $B$ and 8 are actually homeomorphic. I thought they were (though you could think of the 8 as two circles that meet only at one point, in which case they're not homeomorphic).




4. The torus $S^1 \times S^1$ can be thought of as the image of the following map $F\colon \mathbb{R}^2 \to \mathbb{R}^3$:

$$(x, y, z) = F(u, v) = \Big((b + a\cos v)\cos u, (b + a\cos v)\sin u, a\sin v\Big).$$

Verify that $F$ descends to a map of the torus, that $F$ has no self-intersections except those forced by periodicity, and that $DF$ has rank two everywhere. Conclude that the image $F[\mathbb{R}^2]$ is homeomorphic to the torus and hence a manifold.

**Solution:**

First the basic torus we are thinking of is the quotient of $\mathbb{R}^2$ by periodicity in both directions: that is, after the identifications $(0, y)$ $(2\pi, y)$ and $(x, 0)$ $(x, 2\pi)$. As long as $F$ is continuous and periodic, we can think of $F$ as a continuous map on the quotient space. Clearly $F$ is periodic in both $u$ and $v$, since it involves only cosines and sines of $u$ and $v$.

Next we check that it has no other self-intersections. This is in fact only true if $b > a$. $F(u, v) = F(p, q)$ implies that $(b + a\cos v)\cos u = (b + a\cos q)\cos u$, along with $(b + a\cos v)\sin u = (b + a\cos q)\sin p$ and $a\sin v = a\sin q$. We therefore have $|b + a\cos v| = |b + a\cos q|$. If either of these is zero, then we *do* get self-intersections, and they can be zero if $a \geq b$. (If $a = b$ for example then $v = \pi$ and $q = \pi$ map every

$u$ and $p$ to the origin. If $a > b$ then $\sin v = \sin q$ implies $v = q$ or $v = \pi - q$, and in the latter case we can solve $b + a\cos v = b - a\cos v$ for $v$, and we get a self-intersection for every $u$.)

However if $b > a$, then $b + a\cos v$ is positive and so $|b + a\cos v| = |b + a\cos q|$ implies that $b + a\cos v = b + a\cos q$, so that $\cos v = \cos q$. From the last component we see that $\sin v = \sin q$, and therefore $v = q + 2n\pi$ for some $n \in \mathbb{Z}$. And since $b + a\cos v = b + a\cos q \neq 0$, we get that $\cos u = \cos p$ and $\sin u = \sin p$, so that $u = p + 2m\pi$ for some $m \in \mathbb{Z}$. Hence the only self-intersections come from periodicity.

Finally we check the rank of $DF$. We have

$$DF = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} -(b + a\cos v)\sin u & (b + a\cos v)\cos u & 0 \\ -a\sin v\cos u & -a\sin v\sin u & a\cos v \end{bmatrix}.$$

Compute the determinant of the left $2 \times 2$ matrix: we get $a(b + a\cos v)\sin v$ which is nonzero as long as $\sin v \neq 0$, i.e., as long as $v$ is not an integer multiple of $\pi$, since $b + a\cos v$ is never zero. If $v = n\pi$ for some $n \in \mathbb{Z}$, then

$$DF = \begin{bmatrix} -(b \pm a)\sin u & (b \pm a)\cos u & 0 \\ 0 & 0 & \pm a \end{bmatrix},$$

and the determinant of the $2 \times 2$ matrix on the right is $\pm a(b \pm a)\cos u$ which is nonzero as long as $\cos u = 0$. Finally if $\cos u = 0$ then $\sin u = \pm 1$, so that in that case

$$DF = \begin{bmatrix} \mp(b \pm a) & 0 & 0 \\ 0 & 0 & \pm a \end{bmatrix},$$

and the $2 \times 2$ matrix formed by the first and third columns is invertible. Hence in all cases, $DF$ has a $2 \times 2$ submatrix which is invertible, and thus in all cases $DF$ has rank two.

We have established that $F$ is a bijection on the quotient of $\mathbb{R}^2$. There are now two ways to conclude that the quotient of $F$ is a homeomorphism. One is to use the fact that the quotient space $\mathbb{T}^2$ is compact, and thus an invertible continuous map is automatically a homeomorphism. The other is to use the fact that $DF$ has maximal rank, so that $F$ is an open map by the inverse function theorem, which means that its inverse is continuous.

5. Steiner's surface is parametrized as follows. First consider $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

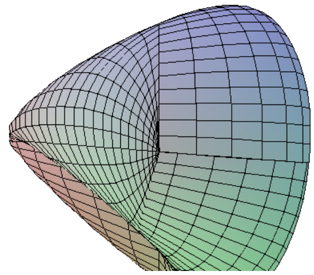
$$F(u, v, w) = (vw, uw, uv).$$

Then Steiner's surface is the image of the 2-sphere $S^2 = \{(u, v, w) \mid u^2 + v^2 + w^2 = 1\}$ under $F$.

Steiner's surface is not a manifold. At what points does it fail to be one?

**Solution:**

We can graph it following the hint: we get something like the figure.



The parametrization is not the most convenient way to think about it though, since it's essentially using spherical coordinates, and we know those are badly behaved at some points (for example, the poles).

There are two questions: what are the self-intersections, and what are the points where the rank is degenerate?

First, we compute $DF$. We get

$$DF = \begin{bmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{bmatrix}.$$

The determinant of this is $\det(DF) = 2uvw$, so $DF$ is nonsingular if all of $u$, $v$, and $w$ are nonzero.

However what we're *really* interested in is not $DF$ but rather $D(F \circ \phi) = (DF)(D\phi)$ where $\phi$ is a coordinate parametrization of the sphere. After all, the Steiner surface is two-dimensional, so the actual coordinate chart will be $F \circ \phi$. (Remember, the Steiner surface is the image of the sphere, not all of $\mathbb{R}^3$.)

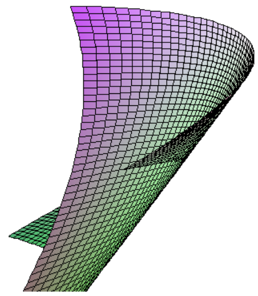
If $DF$ has full rank, then since $D\phi$ has full rank whenever we care, we know that $(DF)(D\phi)$ must be be nondegenerate. The points in question are where $u = 0$, where $v = 0$, or where $w = 0$. In neighborhoods of these points, we need coordinate charts.

Suppose that $u = 0$, without loss of generality (since all three coordinates appear symmetrically). Then $v^2 + w^2 = 1$, so that $DF$ has rank two. The only question is what happens under the parametrization of the sphere. We can either solve for $w$ or for $v$ near a point where $u = 0$; let's suppose it's $w$ without loss of generality. Write the sphere as $(u, v, w) = (s, t, \sqrt{1 - s^2 - t^2})$. Then we get

$$D(F\circ\phi) = (DF)\circ\phi(D\phi) = \begin{bmatrix} 0 & \sqrt{1-s^2-t^2} & t \\ \sqrt{1-s^2-t^2} & 0 & s \\ t & s & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{s}{\sqrt{1-s^2-t^2}} & -\frac{t}{\sqrt{1-s^2-t^2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{st}{\sqrt{1-s^2-t^2}} & \sqrt{1-s^2-t^2} - \frac{t^2}{\sqrt{1-s^2-t^2}} \\ \sqrt{1-s^2-t^2} - \frac{s^2}{\sqrt{1-s^2-t^2}} & -\frac{st}{\sqrt{1-s^2-t^2}} \\ t & s \end{bmatrix}$$

Again computing determinants of submatrices, we see that this matrix is nonsingular except when $s = \pm\frac{sqrt2}{2}$, $t = 0$, or vice versa. At those points it actually has rank one, and we can see it degenerate by plotting a small neighborhood of such points as in the figure below.



Another way this surface fails to be a manifold can be seen by just looking for self-intersections algebraically. It's easy to see that these happen when *two* of $(u, v, w)$ are zero, which corresponds to the poles at $(0, 0, 1)$, $(1, 0, 0$, etc.

6. For what values of $r$ is the surface in $\mathbb{R}^3$ defined by points satisfying

$$x^2 = zy^3 + r$$

a manifold?

**Solution:**

This is an application of the implicit function theorem. Let $F(x, y, z) = x^2 - zy^3$; we want to know whether $r$ is a regular value of $F$.

We compute

$$DF = \begin{bmatrix} 2x & -3y^2z & -y^3 \end{bmatrix},$$

and ask for this to be rank one. This happens as long as either $x \neq 0$ or $y \neq 0$. If both $x = 0$ and $y = 0$, then we have $F(0, 0, z) = 0$, which forces $r$ to be zero.

So as long as $r \neq 0$, we know $r$ is a regular value, so that $F^{-1}(r)$ is a manifold.

If $r = 0$, we can plot $x^2 = zy^3$ and easily see that it is not a manifold.