Math 6230 Homework #1 Solutions

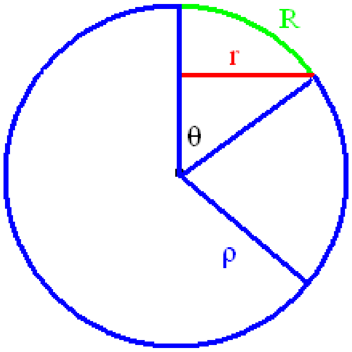
1. Read "Vector Calculus for Differential Geometry" up to Section 4 (Multivariable calculus).

2. Suppose we lived on a small round sphere as in "The Little Prince," and so instead of learning plane geometry in school, we learned spherical geometry since it was much more natural.

   What formulas would replace $C = 2\pi R$ and $A = \pi R^2$ for circumference and area of circles?

   Solution:

   The circumference is easier. Look at the sphere from the side as in Figure 2.



It has some radius ρ. An arc length R can be measured from the north pole (traced out by angle θ), and a circle can be drawn on the sphere by the residents. In Euclidean 3-space that circle will have radius r, although the people on the sphere cannot measure that directly.

From trigonometry we have the relations $R = \rho\theta$ and $\rho \sin\theta = r$. The circumference of the circle is $C = 2\pi r$, and in terms of the observable quantity R and the parameter ρ, the formula is

$$C = 2\pi\rho \sin\left(\frac{R}{\rho}\right).$$

Now to compute the area of the circle, there are two ways to do it. The more direct vector calculus method would be to view it as a surface area problem in three dimensions. We write $z = f(x, y) = \sqrt{\rho^2 - x^2 - y^2}$, and use the surface area formula

$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy = \iint_D \frac{\rho\, dx\, dy}{\sqrt{\rho^2 - x^2 - y^2}} = \int_0^{2\pi} \int_0^r \frac{\rho p\, dp}{\sqrt{\rho^2 - p^2}}\, dp$$

$$= -2\pi\rho\sqrt{\rho^2 - p^2}\Big|_0^r = 2\pi\rho^2 \left[1 - \cos\left(\frac{R}{\rho}\right)\right].$$

The cuter way is to just notice that area is the integral of circumference with respect to the radius, which gives the same answer.

Notice that it's easy to check these formulas in the special cases where we know the answer ($R = \frac{\rho\pi}{2}$ and $R = \rho\pi$). It's also easy to see via Taylor expansions for small R that we get the Euclidean formula back to lowest order.

3. Consider two possible bases for $\mathbb{R}^2$:

$$e_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

vs.

$$f_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

(a) Find the transformation matrix P such that $f_i = \sum_{j=1}^{2} p_i^j e_j$, and the transformation matrix Q such that $e_i = \sum_{j=1}^{2} q_i^j f_j$.

Solution:

We have the equations $f_1 = p_1^1 e_1 + p_1^2 e_2$ and $f_2 = p_2^1 e_1 + p_2^2 e_2$, which translate componentwise into

$$\begin{aligned} -1 &= 2p_1^1 + p_1^2 \\ 1 &= -p_1^1 + 2p_1^2 \\ 2 &= 2p_2^1 + p_2^2 \\ -3 &= -p_1^1 + 2p_2^2 \end{aligned}$$

The solution is $p_1^1 = -\frac{3}{5}$, $p_1^2 = \frac{1}{5}$, $p_2^1 = \frac{7}{5}$, and $p_2^2 = -\frac{4}{5}$. So

$$P = \begin{pmatrix} -\frac{3}{5} & \frac{1}{5} \\ \frac{7}{4} & -\frac{4}{5} \end{pmatrix},$$

and thus the inverse matrix is

$$Q = \begin{pmatrix} -4 & -1 \\ -7 & -3 \end{pmatrix},$$

with $q_1^1 = -4$, $q_1^2 = -1$, $q_2^1 = -7$, and $q_2^2 = -3$.

(b) How would you express the vector $v = 2e_1 - e_2$ in the $\{f\}$-basis?

Solution:

We just convert the e-basis to the f-basis using Q.

$$\begin{aligned} v &= 2e_1 - e_2 \\ &= 2(q_1^1 f_1 + q_1^2 f_2) - (q_2^1 f_1 + q_2^2 f_2) \\ &= 2(-4f_1 - f_2) - (-7f_1 - 3f_2) \\ &= -f_1 + f_2. \end{aligned}$$

(c) Compute explicitly the covectors α^1 and α^2 (satisfying $\alpha^i(e_j) = \delta^i_j$), and the covectors β^1 and β^2 satisfying $\beta^i(f_j) = \delta^i_j$.

Solution:

By inspection we find

$$\alpha^1 = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \end{pmatrix} \quad \text{and} \quad \alpha^2 = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

and

$$\beta^1 = \begin{pmatrix} -3 & -2 \end{pmatrix} \quad \text{and} \quad \beta^2 = \begin{pmatrix} -1 & -1 \end{pmatrix}.$$

4. Suppose $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_n\}$ are two bases for the vector space V, with corresponding dual bases $\{\alpha^1, \cdots, \alpha^n\}$ and $\{\beta^1, \cdots, \beta^n\}$. If $\{e\}$ and $\{f\}$ are related by the formulas

$$f_i = \sum_{j=1}^{n} p^j_i e_j \qquad \text{and} \qquad e_i = \sum_{j=1}^{2} q^j_i f_j.$$

Find the coefficients r^i_j and s^i_j in the formulas

$$\beta^i = \sum_{j=1}^{n} r^i_j \alpha^j \qquad \text{and} \qquad \alpha^i = \sum_{j=1}^{n} s^i_j \beta^j.$$

Check your answer in the situation from Problem 1.

Solution:

To find r^i_j, we plug in the vectors e_k to both sides. We obtain

$$\begin{aligned} \beta^i(e_k) &= \sum_{j=1}^{n} r^i_j \alpha^j(e_k) \\ \beta^i\left(\sum_{j=1}^{n} q^j_k f_j\right) &= \sum_{j=1}^{n} r^i_j \delta^j_k \\ \sum_{j=1}^{n} q^j_k \beta^i(f_j) &= r^i_k \\ q^i_k &= r^i_k. \end{aligned}$$

Similarly we have $s^i_k = p^i_k$.

In the case of Problem 1, we verify that

$$\begin{aligned} \beta^1 &= q^1_1\alpha^1 + q^1_2\alpha^2 = -4\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \end{pmatrix} - 7\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} -3 & -2 \end{pmatrix} \\ \beta^2 &= q^2_1\alpha^1 + q^2_2\alpha^2 = -\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \end{pmatrix} - 3\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} -1 & -1 \end{pmatrix} \\ \alpha^1 &= p^1_1\beta^1 + p^1_2\beta^2 = -\frac{3}{5}\begin{pmatrix} -3 & -2 \end{pmatrix} + \frac{7}{5}\begin{pmatrix} -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \end{pmatrix} \\ \alpha^2 &= p^2_1\beta^1 + p^2_2\beta^2 = \frac{1}{5}\begin{pmatrix} -3 & -2 \end{pmatrix} - \frac{4}{5}\begin{pmatrix} -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \end{pmatrix} \end{aligned}$$

5. In Proposition 2.8, an isomorphism $\zeta\colon (V^*)^* \to V$ given by

$$\zeta(\phi) = \sum_{i=1}^{n} \phi(\alpha^i) e_i$$

was constructed, in terms of a particular basis $\{e_i\}$ and dual basis $\{\alpha^i\}$. Show that for any $\phi \in (V^*)^*$, we get the same vector $\zeta(\phi)$ regardless of which basis we use. In other words, use the transition formulas to check that for any $\phi \in (V^*)^*$, we have

$$\sum_{i=1}^{n} \phi(\alpha^i) e_i = \sum_{i=1}^{n} \phi(\beta^i) f_i.$$

Solution:

This is easy enough, since P and Q are inverses.

$$\begin{aligned}
\sum_{i=1}^{n} \phi(\alpha^i) e_i &= \sum_{i=1}^{n} \phi\left(\sum_{j=1}^{n} s^i_j \beta^j\right) \sum_{k=1}^{n} q^k_i f_k \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} s^i_j q^k_i \phi(\beta^j) f_k \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \left(\sum_{i=1}^{n} p^i_j q^k_i\right) \phi(\beta^j) f_k \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \delta^k_j \phi(\beta^j) f_k \\
&= \sum_{j=1}^{n} \phi(\beta^j) f_j.
\end{aligned}$$

6. Suppose $V = \mathbb{R}^3$, and that $\alpha = -\alpha^1 - \alpha^2 + 2\alpha^3$, $\beta = \alpha^2 + \alpha^3$, and $\gamma = -\alpha^1 + \alpha^2$.

Compute explicitly the 2-forms $\alpha \wedge \beta$ and $\beta \wedge \gamma$. Then compute explicitly the 3-forms $(\alpha \wedge \beta) \wedge \gamma$ and $\alpha \wedge (\beta \wedge \gamma)$. Verify that they are equal.

Solution:

We have by antisymmetry and multilinearity that

$$\begin{aligned}
\alpha \wedge \beta &= (-\alpha^1 - \alpha^2 + 2\alpha^3) \wedge (\alpha^2 + \alpha^3) \\
&= -\alpha^1 \wedge \alpha^2 - \alpha^1 \wedge \alpha^3 - \alpha^2 \wedge \alpha^3 + 2\alpha^3 \wedge \alpha^2 \\
&= -\alpha^1 \wedge \alpha^2 - \alpha^1 \wedge \alpha^3 - 3\alpha^2 \wedge \alpha^3
\end{aligned}$$

and

$$\begin{aligned}
\beta \wedge \gamma &= (\alpha^2 + \alpha^3) \wedge (-\alpha^1 + \alpha^2) \\
&= -\alpha^2 \wedge \alpha^1 - \alpha^3 \wedge \alpha^1 + \alpha^3 \wedge \alpha^2 \\
&= \alpha^1 \wedge \alpha^2 + \alpha^1 \wedge \alpha^3 - \alpha^2 \wedge \alpha^3.
\end{aligned}$$

Thus

$$\begin{aligned}(\alpha \wedge \beta) \wedge \gamma &= \left(-\alpha^1 \wedge \alpha^2 - \alpha^1 \wedge \alpha^3 - 3\alpha^2 \wedge \alpha^3\right) \wedge \left(-\alpha^1 + \alpha^2\right) \\ &= -\alpha^1 \wedge \alpha^3 \wedge \alpha^2 - 3\alpha^2 \wedge \alpha^3 \wedge \alpha^1 \\ &= 4\alpha^1 \wedge \alpha^2 \wedge \alpha^3\end{aligned}$$

and

$$\begin{aligned}\alpha \wedge (\beta \wedge \gamma) &= (-\alpha^1 - \alpha^2 + 2\alpha^3) \wedge \left(\alpha^1 \wedge \alpha^2 + \alpha^1 \wedge \alpha^3 - \alpha^2 \wedge \alpha^3\right) \\ &= 2\alpha^3 \wedge \alpha^1 \wedge \alpha^2 - \alpha^2 \wedge \alpha^1 \wedge \alpha^3 + \alpha^1 \wedge \alpha^2 \wedge \alpha^3 \\ &= 4\alpha^1 \wedge \alpha^2 \wedge \alpha^3.\end{aligned}$$

7. Consider $V = \mathbb{R}^3$ with some basis $\{e_1, e_2, e_3\}$ and dual basis $\{\alpha^1, \alpha^2, \alpha^3\}$, and $W = \mathbb{R}^2$ with some basis $\{f_1, f_2\}$ and dual basis $\{\beta^1, \beta^2\}$. Suppose $T\colon V \to W$ is the linear transformation defined by $T(e_1) = f_1 + 2f_2$, $T(e_2) = f_2$, $T(e_3) = -f_1 + f_2$.

   (a) Compute $T^*\colon W^* \to V^*$ explicitly, i.e., find $T^*(\beta^1)$ and $T^*(\beta^2)$ explicitly in terms of $\{\alpha\}$.

   Solution:

   We have

$$\begin{aligned}T^*(\beta^1)(e_1) &= \beta^1(f_1 + 2f_2) = 1 \\ T^*(\beta^1)(e_2) &= \beta^1(f_2) = 0 \\ T^*(\beta^1)(e_3) &= \beta^1(-f_1 + f_2) = -1, \\ T^*(\beta^2)(e_1) &= \beta^2(f_1 + 2f_2) = 2 \\ T^*(\beta^2)(e_2) &= \beta^2(f_2) = 1 \\ T^*(\beta^2)(e_3) &= \beta^2(-f_1 + f_2) = 1.\end{aligned}$$

   We conclude $T^*(\beta^1) = \alpha^1 - \alpha^3$ and $T^*(\beta^2) = 2\alpha^1 + \alpha^2 + \alpha^3$.

   (b) Compute the 2-form $T^*(\beta^1 \wedge \beta^2)$ on V. (Hint: it will be 2-form on a 3-dimensional space, so it must be

$$T^*(\beta^1 \wedge \beta^2) = a\alpha^1 \wedge \alpha^2 + b\alpha^2 \wedge \alpha^3 + c\alpha^3 \wedge \alpha^1$$

   for some numbers a, b, and c. You need to determine those numbers separately by computing $T^*(\beta^1 \wedge \beta^2)(e_i, e_j)$ for each of the possible pairs $\{e_i, e_j\}$.)

   (c) Compute the 2-form $T^*(\beta^1) \wedge T^*(\beta^2)$ explicitly. Thus verify the product rule

$$T^*(\beta^1 \wedge \beta^2) = T^*(\beta^1) \wedge T^*(\beta^2)$$

   in this case.

Solution: As suggested, we compute

$$\begin{aligned}
T^*(\beta^1 \wedge \beta^2)(e_1, e_2) &= (\beta^1 \wedge \beta^2)\big(T(e_1), T(e_2)\big) \\
&= (\beta^1 \wedge \beta^2)(f_1 + 2f_2, f_2) \\
&= \beta^1(f_1 + 2f_2)\beta^2(f_2) - \beta^1(f_2)\beta^2(f_1 + 2f_2) \\
&= 1, \\
T^*(\beta^1 \wedge \beta^2)(e_1, e_3) &= (\beta^1 \wedge \beta^2)(f_1 + 2f_2, -f_1 + f_2) \\
&= \beta^1(f_1 + 2f_2)\beta^2(-f_1 + f_2) - \beta^1(-f_1 + f_2)\beta^2(f_1 + 2f_2) \\
&= 3, \\
T^*(\beta^1 \wedge \beta^2)(e_2, e_3) &= (\beta^1 \wedge \beta^2)(f_2, -f_1 + f_2) \\
&= \beta^1(f_2)\beta^2(-f_1 + f_2) - \beta^1(-f_1 + f_2)\beta^2(f_2) \\
&= 1.
\end{aligned}$$

Thus we have

$$T^*(\beta^1 \wedge \beta^2) = \alpha^1 \wedge \alpha^2 + 3\alpha^1 \wedge \alpha^3 + \alpha^2 \wedge \alpha^3.$$