Math 6230 Midterm Exam
Due Wednesday, October 21 in class

- **Do any four out of the five problems.**
- You may use only class notes (Vector Calculus for Differential Geometry) and homework solutions.
- You may not consult with anyone except me.
- You may not use computer algebra software; show all work.

1. The *Hopf fibration* is a map $F\colon S^3 \to S^2$ defined in the following way. We first define $\tilde{F}\colon \mathbb{R}^4 \to \mathbb{R}^3$ by

$$\tilde{F}(w,x,y,z) = (w^2 + x^2 - y^2 - z^2, 2(wz + xy), 2(xz - wy)),$$

and then verify that if $w^2 + x^2 + y^2 + z^2 = 1$, then $\tilde{F}(w,x,y,z)$ is actually in S^2.

(a) Verify that if $\mathbf{x} \in S^3$, then $\tilde{F}(\mathbf{x}) \in S^2$, so that the restriction of $\tilde{F}$ actually is a smooth map from S^3 to S^2.

Solution:

$$\begin{aligned}
|\tilde{F}(w,x,y,z)|^2 &= (w^2 + x^2 - y^2 - z^2)^2 + 4(wz + xy)^2 + 4(xz - wy)^2 \\
&= (w^2 + x^2)^2 + (y^2 + z^2)^2 - 2(w^2 + x^2)(y^2 + z^2) \\
&\qquad + 4w^2z^2 + 8wxyz + 4x^2y^2 + 4x^2z^2 \\
&\qquad - 8wxyz + 4w^2y^2 \\
&= (w^2 + x^2)^2 + (y^2 + z^2)^2 + 2(w^2 + x^2)(y^2 + z^2) \\
&= (w^2 + x^2 + y^2 + z^2)^2,
\end{aligned}$$

so that $|(w,x,y,z)| = 1$ implies $|\tilde{F}(w,x,y,z)| = 1$.

Since $\tilde{F}$ is clearly smooth on $\mathbb{R}^4$ (as a polynomial in the variables) and the embedding $\iota_3\ colon S^3 \to \mathbb{R}^4$ is smooth, and the embedding $\iota_2\colon S^2 \to \mathbb{R}^3$ is smooth with smooth inverse $\iota_2^{-1}\colon \iota_2[S^2] \subset \mathbb{R}^3 \to S^2$, we know $F = \iota_2^{-1} \circ \tilde{F} \circ \iota_3$ is smooth as a map from S^3 to S^2.

(b) Verify that F has rank two everywhere on S^3.

Solution:

We want to show that $(F_*)_p\colon T_pS^3 \to T_{F(p)}S^2$ has maximal rank. One way to do this is to use coordinates and do it directly, although that's kind of painful. A simpler approach is to use the three vector fields V_1, V_2, V_3 that span T_pS^3. We have

$$D\tilde{F} = \begin{pmatrix} 2w & 2x & -2y & -2z \\ 2z & 2y & 2x & 2w \\ -2y & 2z & -2w & 2x \end{pmatrix}.$$

so that (with (t,u,v) the coordinates on the range $\tilde{F}[\mathbb{R}^4]$)

$$\begin{aligned}
F_*(V_1) &= F_*\left(-x\,\tfrac{\partial}{\partial w} + w\,\tfrac{\partial}{\partial x} - z\,\tfrac{\partial}{\partial y} + y\,\tfrac{\partial}{\partial z}\right) \\
&= 4(-xz + yw)\,\tfrac{\partial}{\partial u} + 4(xy + wz)\,\tfrac{\partial}{\partial v} \\
F_*(V_2) &= F_*\left(-y\,\tfrac{\partial}{\partial w} + z\,\tfrac{\partial}{\partial x} + w\,\tfrac{\partial}{\partial y} - x\,\tfrac{\partial}{\partial z}\right) \\
&= 4(xz - yw)\,\tfrac{\partial}{\partial t} + 2(y^2 + z^2 - w^2 - x^2)\,\tfrac{\partial}{\partial v} \\
F_*(V_3) &= F_*\left(-z\,\tfrac{\partial}{\partial w} - y\,\tfrac{\partial}{\partial x} + x\,\tfrac{\partial}{\partial y} + w\,\tfrac{\partial}{\partial z}\right) \\
&= -4(xy + wz)\,\tfrac{\partial}{\partial t} - 2(y^2 + z^2 - w^2 - x^2)\,\tfrac{\partial}{\partial u}.
\end{aligned}$$

Since $t = w^2 + x^2 - y^2 - z^2$, $u = 2(wz + xy)$, and $v = 2(xz - wy)$, we get the three vectors

$$\begin{aligned}
F_*(V_1) &= -2v\,\tfrac{\partial}{\partial u} + 2u\,\tfrac{\partial}{\partial v} \\
F_*(V_2) &= 2v\,\tfrac{\partial}{\partial t} - 2t\,\tfrac{\partial}{\partial v} \\
F_*(V_3) &= -2u\,\tfrac{\partial}{\partial t} + 2t\,\tfrac{\partial}{\partial u}.
\end{aligned}$$

Now the 2-sphere is of course described by $t^2 + u^2 + v^2 = 1$. If $v \neq 0$ then $F_*(V_1)$ and $F_*(V_2)$ are linearly independent; if $u \neq 0$ then $F_*(V_1)$ and $F_*(V_3)$ are linearly independent; if $t \neq 0$ then $F_*(V_2)$ and $F_*(V_3)$ are linearly independent. At least one of (t,u,v) is nonzero, so the span of $\{F_*(V_1), F_*(V_2), F_*(V_3)\}$ is always two-dimensional, and hence it is all of $T_{F(p)}S^2$.

(c) Show based on your result above that $F^{-1}(p)$ is diffeomorphic to S^1 for each $p \in S^2$. (Hint: you can see that $F^{-1}(p)$ is a one-dimensional manifold, and you can also show that it's a closed subset. What does that tell you?)

Solution:

Since F_* has maximal rank everywhere, by the implicit function theorem for manifolds we know that $F^{-1}(q)$ is a manifold, and its dimension must be $3-2=1$. Also since F is continuous, $F^{-1}(q)$ is the inverse image of the closed set $\{q\}$ and so it's also closed. A closed subset of a compact space is also compact, so $F^{-1}(q)$ is a compact one-dimensional manifold, and hence it has to be S^1.

2. Consider $\mathbb{R}^{n^2}$ as the set of $n \times n$ matrices, and consider $\mathbb{R}^{n(n+1)/2}$ as the set of symmetric $n \times n$ matrices.

(a) Define $H\colon \mathbb{R}^{n^2} \to \mathbb{R}^{n(n+1)/2}$ by $H(A) = A^T A$. If I_n denotes the $n \times n$ identity matrix, prove that I_n is a regular value of H, so that $H^{-1}(I_n) = O(n)$ is a manifold of dimension $n(n-1)/2$ (as was claimed in the notes).

Solution:

Let's take the simple case when $n = 2$. Then with the matrix $A = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ we have

$$A^T A = \begin{pmatrix} t & u \\ u & v \end{pmatrix} = \begin{pmatrix} w^2 + y^2 & wx + yz \\ wx + yz & x^2 + z^2 \end{pmatrix}.$$

Therefore

$$H(w, x, y, z) = (w^2 + y^2, wx + yz, x^2 + z^2).$$

We have

$$DH(w, x, y, z) = \begin{pmatrix} 2w & 0 & 2y & 0 \\ 2x & 2w & 2z & 2y \\ 0 & 2x & 0 & 2z \end{pmatrix}.$$

Now when $A^T A = I_n$ we have $w^2 + y^2 = 1$, $wx + yz = 0$, and $x^2 + z^2 = 1$. Computing the dot product of the rows of DH we see that the first and third are obviously orthogonal, while the first and second are orthogonal since $wx+yz = 0$, and the second and third are orthogonal since $wx + yz = 0$.

Hence we have three orthogonal rows of DH, which means the rows are linearly independent and the rank of DH is three.

More generally, we write $A = (a_{ij})$ and $A^T A = (b_{ij})$, so that

$$b_{ij} = \sum_{k=1}^{n} a_{ki} a_{kj}$$

for $i \leq j$. The b_{ij} form $\frac{n(n+1)}{2}$ functions of the n^2 variables a_{ij}, and the rows of DH correspond to the gradients of these functions. We have

$$\nabla b_{ij} = \sum_{k=1}^{n} a_{ki} \tfrac{\partial}{\partial a_{kj}} + \sum_{k=1}^{n} a_{kj} \tfrac{\partial}{\partial a_{kj}},$$

and so the dot products of these are

$$\begin{aligned} \langle \nabla b_{ij}, \nabla b_{pq} \rangle &= \sum_{k=1}^{n} \sum_{r=1}^{n} a_{ki} a_{rp} \delta_{kr} \delta_{qj} + a_{kj} a_{rp} \delta_{kr} \delta_{qi} + a_{ki} a_{rq} \delta_{kr} \delta_{jp} + a_{kj} a_{rq} \delta_{kr} \delta_{ip} \\ &= n \sum_{k=1}^{n} (a_{ki} a_{kp} \delta_{qj} + \delta_{qi} a_{kj} a_{kp} + \delta_{jp} a_{ki} a_{kq} + a_{kj} a_{kq} \delta_{ip}). \end{aligned}$$

Now whenever $A^T A = I_n$, we will have

$$\sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij},$$

so that

$$\langle \nabla b_{ij}, \nabla b_{pq} \rangle = 2n(\delta_{ip} \delta_{qj} + \delta_{qi} \delta_{jp}).$$

The only way to get anything nonzero is if either $i = p$ and $j = q$, or $i = q$ and $j = p$. Since we might as well have $i \leq j$ and $p \leq q$, we conclude these are orthogonal to each other. Hence in particular they are $\frac{n(n+1)}{2}$ linearly independent vectors forming the rows of DH, and hence DH has rank $\frac{n(n+1)}{2}$.

(b) What is the tangent space to $O(n)$ at the identity I_n (as a subset of the vector space of all matrices)?

Solution: Consider a curve $A(t)$ of matrices in $O(n)$. Then for each t we have $A(t)^T A(t) = I_n$, and differentiating with respect to t we get

$$\frac{dA^T}{dt} A + A^T \frac{dA}{dt} = 0_n.$$

If $A(0) = I_n$ and $A'(0) = V$, this equation implies that

$$V^T + V = 0_n,$$

so that V is an antisymmetric matrix.

Conversely if V is an antisymmetric matrix, then the curve $A(t) = e^{tV}$ satisfies

$$A(t)^T A(t) = e^{tV^T} e^{tV} = e^{-tV} e^{tV} = I_n,$$

so that every antisymmetric matrix is actually a tangent vector to an orthogonal matrix.

3. Let G be an n-dimensional Lie group with identity e. For any fixed $g \in G$, let $L_g \colon G \to G$ denote the left-translation map given by $L_g(h) = g \cdot h$. Prove that $((L_g)_*)_e \colon T_eG \to T_gG$ is an isomorphism of vector spaces for any g.

Given any vector $v_0 \in T_eG$, define a vector field V on G by the formula $V(g) = ((L_g)_*)_e(v_0)$. (Such a vector field is called "left-invariant.") Use left-invariant vector fields to prove that the tangent bundle of a Lie group is always trivial.

Solution: Since G is a Lie group, every element g has an inverse element g^{-1}. Furthermore $L_{g^{-1}}$ is the inverse of L_g, since for any $h \in G$,

$$L_{g^{-1}} L_g(h) = g^{-1}(gh) = (g^{-1}g)h = eh = h.$$

By the chain rule we conclude

$$(L_{g^{-1}})_*(L_g)_* = (L_{g^{-1}} \circ L_g)_* = (\mathrm{id})_* = \mathrm{id}$$

is the identity operator on tangent spaces. Since $(L_g)_*$ is invertible for every tangent space, it is an isomorphism.

Now T_eG is an n-dimensional vector space, so we can find a basis $\{e_1, \cdots, e_n\}$ of T_eG. Define $E_k(g) = (L_g)_*(e_k)$ for $1 \le k \le n$. Since $(L_g)_*$ is an isomorphism, the vectors $E_1(g), \cdots, E_n(g)$ are all linearly independent for each g. And these vector fields are also smooth in g since L_g is.

So we can just use the homework result. Or more specifically, we construct the map $\varphi \colon G \times \mathbb{R}^n \to TG$ as

$$\varphi(g, v^1, \cdots, v^n) = \sum_{k=1}^{n} v^k (L_g)_*(e_k),$$

and observe that this map is smooth, invertible, and an isomorphism on each vector space. Hence TG is trivial.

4. Consider the Lie group A consisting of all invertible linear maps, $u \mapsto pu + q$ for $p > 0$, with the group operation consisting of composition of maps. (Note that u is just a dummy variable; the set of objects we're concerned about is parametrized by coordinates (p, q).) The identity element e of course corresponds to the identity linear map, $p = 1$ and $q = 0$.

   (a) Show that $A = \{(p, q) \,|\, p > 0\} \subset \mathbb{R}^2$ is isomorphic to the matrix subgroup consisting of $\left\{ \left(\begin{smallmatrix} 1 & q \\ 0 & p \end{smallmatrix} \right) \,|\, q \in \mathbb{R}, p > 0 \right\}$. That is, show that multiplication in the matrix subgroup corresponds to composition in the group of linear functions.

   Solution:

   Suppose we have elements $g = (p, q)$ and $h = (r, s)$. Then for any dummy variable u, we have

$$gh(u) = g(ru + s) = p(ru + s) + q = pru + (ps + q).$$

   So the group operation is $(p, q) \cdot (r, s) = (pr, ps + q)$.

   The group operation on matrices is

$$\begin{pmatrix} 1 & q \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & r \end{pmatrix} = \begin{pmatrix} 1 & s + rq \\ 0 & pr \end{pmatrix},$$

   which corresponds to $(p, q) \cdot (r, s) = (pr, s + rq)$, which is actually backwards.

   (b) Compute the left-translation maps $L_{(a,b)}$ in coordinates, and from this compute the differentials $(L_{(a,b)})_*$ from T_eG to $T_{(a,b)}G$ in coordinates.

   Solution:

   Consider coordinates (p, q) on the domain and (r, s) on the range, so that the map is

$$(r, s) = (a, b) \cdot (p, q) = (ap, aq + b).$$

   Then the differential is

$$\begin{aligned} (L_{(a,b)})_* \left(\frac{\partial}{\partial p}\Big|_{(p,q)} \right) &= \frac{\partial r}{\partial p} \frac{\partial}{\partial r}\Big|_{(r,s)} + \frac{\partial s}{\partial p} \frac{\partial}{\partial s}\Big|_{(r,s)} \\ &= a \frac{\partial}{\partial r}\Big|_{(r,s)} \\ (L_{(a,b)})_*(\left(\frac{\partial}{\partial q}\Big|_{(p,q)} \right) &= a \frac{\partial}{\partial s}\Big|_{(r,s)}. \end{aligned}$$

   (c) For the basis vectors $u_0 = \frac{\partial}{\partial p}\big|_e$ and $v_0 = \frac{\partial}{\partial q}\big|_e$, compute the left-invariant vector fields U and V explicitly.

   Solution:

   We already computed the general formula, so now we just change variables, using $(r, s) = (a, b) = (p, q)$:

$$U(p, q) = p \frac{\partial}{\partial p}\Big|_{(p,q)}$$

   and

$$V(p, q) = p \frac{\partial}{\partial q}\Big|_{(p,q)}.$$

5. Consider a hexagon with edges identified by the rule $abc^{-1}dbc$. (That is, the sides a and d are *not* identified with any other side.)

   Is the resulting quotient space a manifold-with-boundary? If so, identify it using the classification of surfaces with boundary. If not, explain why not.

   Solution:

   The only points at which it's not obviously a manifold are the edges a and d, and the only points where those aren't obviously boundary points of a manifold-with-boundary are the endpoints of a and the endpoints of d.

   Let's say the beginning of a (followed counterclockwise) is P and the end of a is Q. Then Q is the beginning of b, so it's also the end of d. P is the end of c, so it's also the end of b. Since it's the end of b, it's the beginning of c. So inserting the points explicitly, the diagram is

$$PaQbPc^{-1}PdQbPcP.$$

   Hence the two segments a and d are really just one circle, from P to Q along a and back along d backwards to P. To see this explicitly, let's get rid of the point Q by cutting across from the point where a and c meet, to the point where d and c meet. We'll call this new edge e.

   Flip the top half and glue it back along the bottom on b, and we get the polygon $ecceda^{-1}$, and we can rename the side da^{-1} to f to get the polygon $eccef$.

   The start and end of f is the same point P, so f is a circle. Thus if we glue in a disc along f, we can remove it entirely (we just have to remember to take it out at the end.

   So after gluing in a disc we get the square $ecce = eecc$. This is the connected sum of two projective planes $\mathbb{P}^2 \# \mathbb{P}^2$, which is a Klein bottle.

   Thus the original hexagon is the connected sum of two projective planes with a disc removed.