

Math 4650 Sample Final

1. Suppose you are given the method

$$w_{i+1} = w_{i-1} + h(af(t_i, w_i) + bf(t_{i-1}, w_{i-1}))$$

for approximating the differential equation $y'(t) = f(t, y(t))$.

- (a) What should a and b be so that the local truncation error is as small as possible? What will the local truncation error be in that case?
 - (b) Is the method stable? Strongly stable?
 - (c) Is the method convergent? How do you know?
2. Use Romberg integration to estimate $\int_0^2 x^2 dx$. How large does k have to be before $R_{k,k}$ is the exact answer?
3. For the differential equation $y' = 2y - 4t$ with $y(0) = 1$, verify that the exact solution is $y(t) = 2t + 1$.
- (a) Use Euler's method with step size $h = \frac{1}{2}$ to estimate $y(1)$.
 - (b) Use Heun's method with step size $h = 1$ to estimate $y(1)$.
 - (c) Use the second-order Taylor method with step size $h = 1$ to estimate $y(1)$.

Which method comes closest to the exact solution?

4. (a) Suppose $y'(t) = f(t, y)$ is some differential equation with initial condition $y(0) = \alpha$ given. If Euler's method with step size h is used to get w_1 , an approximation to $y(h)$, how far off do you expect it to be?
- (b) Now suppose Euler's method with step size $h/2$ is used to get \tilde{w}_2 , also an approximation to $y(h)$. What error do you expect using this method?
- (c) Show how to estimate the true value of $y(h)$ from the computed values w_1 and \tilde{w}_2 .
- (d) Explain briefly how you could use this to derive an adaptive Euler's method.
5. Give the LU -decomposition of the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Then solve the equation $A\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ using the decomposition and back-substitution.

6. Suppose you want to solve an equation involving an $n \times n$ matrix of bandwidth 4 which looks like the following:

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & \cdots \\ a_{21} & a_{22} & a_{23} & 0 & 0 & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & \cdots \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & \cdots \\ 0 & 0 & a_{53} & a_{54} & a_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{pmatrix}$$

Write an algorithm, generalizing Crout's algorithm, to solve this system. How does the number of operations in your algorithm depend on n ?

7. The STORMEY algorithm (Second/Third Order Runge-Moulton Egg Yolks) is defined as follows:

$$w_{i+1} = w_i + \frac{h}{4}f(t_i + h, w_{i+1}) + \frac{3h}{4}f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3}f(t_i, w_i)\right).$$

For the differential equation $y'(t) = 2y(t) - t$, work out what precisely the STORMEY algorithm does. For $h = 1$ and $y(0) = 1$, what is the estimate for $y(2)$?