Math 4650 Sample Midterm

1. Derive, using Taylor series, the best possible algorithm for computing $f'(x_o)$ if you are given $f(x_o)$, $f(x_o+h)$, and $f(x_o+3h)$. What is the first term in the error?

   Solution: We have three points, which means we will be able to cancel out the first three terms of the Taylor series, while the fourth term will produce the error.

   So we write

$$\begin{aligned} f(x_o) &= f(x_o) \\ f(x_o+h) &= f(x_o) + hf'(x_o) + \tfrac{f''(x_o)}{2}h^2 + \tfrac{f'''(x_o)}{6}h^3 + \cdots \\ f(x_o+3h) &= f(x_o) + 3hf'(x_o) + \tfrac{9f''(x_o)}{2}h^2 + \tfrac{9f'''(x_o)}{2}h^3 + \cdots \end{aligned}$$

   The first equation is trivial. If we subtract the third equation from 9 times the second equation (to eliminate the $f''(x_o)$ term), we get

$$9f(x_o+h) - f(x_o+3h) = 8f(x_o) + 6hf'(x_o) - 3h^3 f'''(x_o) + \cdots,$$

   so that

$$f'(x_o) = \frac{9f(x_o+h) - 8f(x_o) - f(x_o+3h)}{6h} + \frac{h^2}{2}f'''(x_o) + \cdots$$

2. What does the following algorithm compute? Write the output for $N = 3$. (Don't simplify!) Then write a more efficient version of the algorithm.

```
Y=1
P=1
FOR K FROM 1 TO N DO
   P=1
   FOR J FROM 1 to K DO
      P=P*J
   END DO
   Y=Y+1/P
END DO
PRINT Y
```

   Solution: The inner loop clearly computes $K!$, while the outer loop computes

$$Y = 1 + 1/1! + 1/2! + 1/3! + \cdots + 1/N!$$

   So basically this algorithm computes an approximation to e. When $N = 3$, the output is $Y = 1 + 1 + 1/2 + 1/6$.

   Observing, as in the first homework assignment, that

$$Y = 1+1/1!+1/2!+\cdots+1/N! = 1+1+1/2*(1+1/3*(...1+1/(N-1)*(1+1/N))...),$$

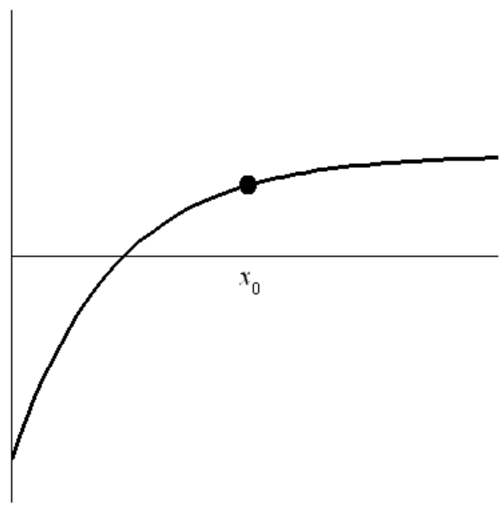
   we can rewrite the algorithm as

```
Y=1
FOR K FROM 1 TO N DO
   Y=1+Y/(N+1-K)
END DO
PRINT Y
```

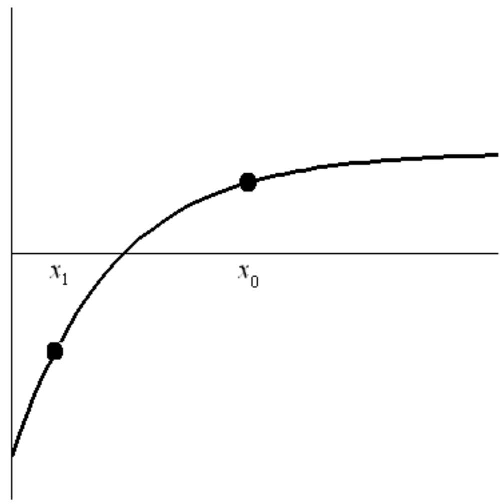
When $N = 3$, we can check the algorithm produces $Y = 1$ before starting, $Y = 1 + 1/3$ on the first step, $Y = 1 + 1/2 + 1/6$ on the second step, and $Y = 1 + 1 + 1/2 + 1/6$ on the third step, as it should.

The old algorithm involved about $N^2/2 + 2N$ computations, while the new one involves about $2N$ computations. It also adds smaller numbers together first, which reduces roundoff error.

3. If we run 8 steps of Newton's method on the following function, with initial condition as shown, will the estimate for the root be higher or lower than the actual root? Why?



If we run 12 steps of the secant method on the following function, with initial conditions as shown, will the estimate for the root be higher or lower than the actual root? Why?



Solution: From the picture, we see that the graph is concave down. This implies that every tangent line drawn to the graph will be strictly *above* the graph. Hence every tangent line will cross the graph at a point *smaller* than the actual root. As a result, no matter how many iterations of Newton's method we use, the final estimate will always be smaller than the actual root.

On the other hand, the graph being concave down implies that every secant line is *below* the curve between the points of interest and *above* the curve outside that interval. If we start with two points that bound the root, and draw the secant line between them, the approximation they give will always be above the actual root. If on the other hand we start with two points above the actual root, the secant line between them will cross the axis below the actual root.

Hence this is a bit more complicated. Clearly if p is the root, $x_o > p$ and $x_1 < p$. At the next step, $x_2 > p$. Since x_1 and x_2 still bound the root, $x_3 > p$. Now x_2 and x_3 are both above the root, so $x_4 < p$. In general, the pattern is

$$x_0 > p, x_1 < p, x_2 > p, x_3 > p, x_4 < p, x_5 > p, x_6 > p, ...$$

Hence x_{12} is also greater than the actual root p, while x_{13} is less than the actual root.

4. A sequence is generated with $x_0 = 1$, $x_n = \sqrt{6 + x_{n-1}}$.
   (a) What is the limit p of the sequence?
   (b) What are the order of convergence and the asymptotic error constant?

   Solution:

   (a) The sequence is of the form $x_n = g(x_{n-1})$. If the limit p exists, then it must be a fixed point, so that $p = g(p)$. Solving this equation, we get

$$\begin{aligned} p &= \sqrt{6+p} \\ p^2 &= 6+p \\ p^2 - p - 6 &= 0 \\ (p-3)(p+2) &= 0. \end{aligned}$$

So $p = 3$ or $p = -2$. We throw away $p = -2$ since it doesn't satisfy the original equation $p = \sqrt{6+p}$. So the limit must be $p = 3$.

(b) Now to see whether it converges and how fast, we compute $g'(3) = \frac{1}{2\sqrt{6+3}} = \frac{1}{6}$. Since $g'(3) \neq 0$, the sequence converges linearly (i.e., with $\alpha = 1$) and with asymptotic error constant $\lambda = |\frac{1}{6}| = \frac{1}{6}$.

5. Suppose you have values $\ln 1 = 0$, $\ln 2 = 0.693$, and $\ln 3 = 1.099$. Give a formula for an interpolation estimate of $\ln 2.5$ (don't multiply it out), and estimate how large the error could be.

Solution:

The only interpolating polynomial we can use is a quadratic. The Lagrange polynomial is easy to write down (especially since we don't have to actually evaluate it). It predicts

$$\begin{aligned}\ln 2.5 &= \ln 1 \frac{(2.5-2)(2.5-3)}{(1-2)(1-3)} + \ln 2 \frac{(2.5-1)(2.5-3)}{(2-1)(2-3)} + \ln 3 \frac{(2.5-1)(2.5-2)}{(3-1)(3-2)} \\ &= 0.693 \cdot 0.75 + 1.099 \cdot 0.375.\end{aligned}$$

Now the error estimate comes from the general error formula

$$f(x) - P_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-1)(x-2)(x-3).$$

Hence the error is at most

$$E(2.5) \leq \frac{|f'''(\xi)|}{6}|(2.5-1)(2.5-2)(2.5-3)| = \frac{|f'''(\xi)|}{16},$$

for some $\xi \in [1, 3]$.

Since $f(x) = \ln x$, we have $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, and $f'''(x) = 2x^{-3}$. The maximum of this on $[1, 3]$ is 2. Hence the error is at most

$$E(2.5) \leq \frac{2}{16} = \frac{1}{8} = 0.125.$$